UNCLASSIFIED

AD NUMBER
ADB021180
LIMITATION CHANGES
TO: **Approved for public release; distribution is unlimited.**
FROM: **Distribution authorized to U.S. Gov't. agencies only; Test and Evaluation; FEB 1976. Other requests shall be referred to Air Force Flight Dynamics Lab., Wright-Patterson AFB, OH 45433.**
AUTHORITY
AFFDL ltr 2 May 1979

THIS PAGE IS UNCLASSIFIED

2

AFFDL-TR-76-11
VOLUME I

ADB021180

L

STRESS ANALYSIS OF AIRCRAFT TIRES
Volume I. Analytical Formulation

MATHEMATICAL SCIENCES NORTHWEST, INC.
P. O. BOX 1887
BELLEVUE, WASHINGTON 98009

FEBRUARY 1976

TECHNICAL REPORT AFFDL-TR-76-11, VOLUME I
Final Report for Period June 1974 – January 1976

Distribution is limited to U.S. Government agencies; Test and Evaluation; statement applied February 1976. Other requests for this document must be referred to AF Flight Dynamics Laboratories, AFFDL/FEM, WPAFB, Ohio 45433.

AIR FORCE FLIGHT DYNAMICS LABORATORY
AIR FORCE WRIGHT AERONAUTICAL LABORATORIES
AIR FORCE SYSTEM COMMAND
WRIGHT-PATTERSON AIR FORCE BASE, OHIO 45433

NOTICE

When Government drawings, specifications, or other data are used for any purpose other than in connection with a definitely related Government procurement operation, the United States Government thereby incurs no responsibility nor any obligation whatsoever; and the fact that the government may have formulated, furnished, or in any way supplied the said drawings, specifications, or other data, is not to be regarded by implication or otherwise as in any manner licensing the holder or any other person or corporation, or conveying any rights or permission to manufacture, use, or sell any patented invention that may in any way be related thereto.

This technical report has been reviewed and is approved.

James R. Hampton

JAMES R. HAMPTON
Project Engineer

Howell K. Brewer

HOWELL K. BREWER
Chief, Mechanical Branch
Vehicle Equipment Division

FOR THE COMMANDER

Ambrose B. Nutt

AMBROSE B. NUTT, Director
Vehicle Equipment Division
Air Force Flight Dynamics Laboratory

Copies of this report should not be returned unless return is required by security considerations, contractual obligations, or notice on a specific document.

AIR FORCE/56780/25 August 1977 — 100

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

19 REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER (18) AFFDL-TR-76-11-VOLUME I -1	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) (6) STRESS ANALYSIS OF AIRCRAFT TIRES. Volume I. Analytical Formulations.		5. TYPE OF REPORT & PERIOD COVERED (9) Final Report, 24 June [redacted]74-8 Jan[redacted] [redacted]76
		6. PERFORMING ORG. REPORT NUMBER MSN-N-75-317-1 (14)
7. AUTHOR(s) (10) Alexander L. Deak		8. CONTRACT OR GRANT NUMBER(s) (15) F33615-74-C-3102
9. PERFORMING ORGANIZATION NAME AND ADDRESS Mathematical Sciences Northwest, Inc. P. O. Box 1887, Bellevue, WA 98009		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS (16) Project 1369 (17) Task 01
11. CONTROLLING OFFICE NAME AND ADDRESS Air Force Flight Dynamics Laboratory Air Force Systems Command Wright-Patterson Air Force Base, Ohio		12. REPORT DATE (11) Feb[redacted] [redacted]76
		13. NUMBER OF PAGES 125
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office) (12) 116p.		15. SECURITY CLASS. (of this report) Unclassified
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Distribution is limited to U. S. Government agencies; Test and Evaluation; Statement applied February 1976. Other requests for this document must be referred to AF Flight Dynamics Laboratories, AFFDL/FEM, WPAFB, Ohio 45433.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

variational methods
tires
laminated shells
nonlinear analysis
finite elements

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

Presented is the nonlinear finite element analysis of multi-layered aircraft tires. A piecewise parabolic shear stress distribution is introduced to cope with arbitrary stress distribution across the thickness coordinate. Each layer is an orthotropic ply consisting of cords embedded in an elastic rubber matrix. Triangular plate elements are used with five generalized displacements at the nodes. The computer code is written for the CDC-6600 machine under the Scope 3.3 and 3.4.3 operating systems.

404058

1B

FOREWORD

This report documents the work accomplished by Mathematical Sciences Northwest, P. O. Box 1887, Bellevue, Washington, under Contract No. F33615-74-C-3102, under Project No. 1369, Task No. 01. Volume II contains the User's Manual.

This work was sponsored by the Air Force Flight Dynamics Laboratory, Wright-Patterson Air Force Base, Dayton, Ohio. Mr. J. R. Hampton (AFFDL/FEM) was the technical monitor for the Air Force.

This report covers the research conducted from 24 June 1974 through 8 January 1976. The manuscript was released by the author in February 1976 for publication as a technical report.

TABLE OF CONTENTS

TABLE OF CONTENTS (Concluded)

LIST OF ILLUSTRATIONS

LIST OF TABLES

SECTION I

INTRODUCTION

In the static analysis of continua by the finite element method, the stiffness and load matrices are obtained by the application of the principle of stationary potential energy within the displacement method. In this approach, the strain energy of an element is expressed in terms of some assumed displacement functions. For the monotonic convergence to the actual solution, it is necessary that these functions should satisfy the compatibility conditions at the boundaries of the elements. The construction of displacement functions for the interior of an element which will also ensure compatibility at the boundaries, in most cases, is rather complicated even for elements with simple geometry.

In 1964, Pian [1] proposed an approach for the static analysis of continua, by means of which compatible stiffness matrices could be obtained without undue difficulties. In Pian's approach [2,3,4], one assumes stresses within the element and displacements on its boundary, hence the word hybrid. This approach is extended for the nonlinear analysis of multilayered shell structures with shear deformation [5]. It contains the following new developments in finite element formulations:

(a) the foundation of the hybrid stress finite element method for initial stress problems [6],

(b) the adoption of the above initial stress theory for shell-type structures, and

(c) the extension of the resulting shell theory for the analysis of laminated shells with arbitrary stress distribution across the thickness coordinate.

Since the ultimate goal of this analysis is the solution of the unbonded contact of the tire with a rigid frictionless surface, flat triangular reference surface elements are used. The coupling of these flat plate elements to form a discretized shell reference surface is accomplished by the nodal displacements referred to a fixed global reference system, which is chosen to be the base vectors of the toroidal reference surface of pneumatic tires. Three rectilinear displacements and two rotations are assigned to each node of the triangular surface element. Thus, each element is characterized by 15 generalized nodal displacements. These surface elements are built up of layers of different elastic properties and cord angle orientations with respect to the local reference frame of the element under consideration.

Within the hybrid stress method, an equilibrating stress field is assumed over each layer in terms of simple polynomials whose coefficients are then related to the aforementioned generalized displacements by the hybrid stress variational principle. Thus, a shell theory results.

An important aspect of the finite element concept is that the finite elements may first be considered to be disjoint for the purpose of approximating a function locally over an element. That is, one can consider an individual element to be completely isolated from the collection of all elements and can proceed to approximate a function over the element in terms of its values at the nodes of the element, independent of the ultimate location of the element in the connected model and independent of the behavior of the function in other finite elements. Thus, it is possible to develop

a catalog of the various finite elements in which nodal values of the local approximation are left arbitrary.

Clearly, in order to stay within the philosophy of the finite element analysis, the aforementioned equilibrating layer stress fields must be disjoint from layer to layer. Moreover, this local (layer) stress approximation must be the same over each layer, usually simple polynomials whose coefficients will be referred to as stress coordinates. Recall now that in the usual finite element method, the connectivity of disjoint elements are established by relating the local coordinates to global coordinates, often called as the assembling procedure. In this case, the connectivity of the layers within a finite element is established by satisfying the interlayer equilibrium, which is facilitated by the matching of appropriate stress coordinates of the two layers under consideration.

Since, in the hybrid stress method, the interelement equilibrium need not be satisfied explicitly, the set of stress coordinates are independent for each element. Connectivity is satisfied in the variational sense. Of course, the price one pays for the element-wise disjoint stress field is that, in the construction of the element stiffness matrix, one must eliminate the stress coordinates in terms of the generalized displacements in the hybrid stress functional, in order to construct the stiffness matrix associated with the generalized nodal displacements.

The extension to nonlinear problems utilizes an intrinsic local coordinate frame which follows the element during deformation. The associated initial stress resultants and the effect of the displacements on the equilibrium equations manifest themselves in the particular solution of the element equilibrium equations in terms of the stress resultants.

The initial or unloaded geometry of the carcass is represented by an arbitrary plane curve which, when rotated about the wheel axis, defines the tire middle surface. The carcass thickness varies in the meridian direction but is constant in the circumferential direction. The initial cord angle varies with the meridian position according to the classical lift equation of bias tire construction [7].

The tire carcass material is treated elastically as a laminated anisotropic composite. Each ply is considered as an orthotropic layer consisting of elastic textile cords embedded in an elastic rubber matrix. The principle orthotropic moduli for a given ply are then calculated in terms of the elastic properties of the cord and rubber based on the principle of compatible deformation.

The external loads applied to the tire are inflation pressure and inertial forces due to rotation.

In the numerical implementation of the aforementioned disjoint stress field concept, certain software problems associated with the construction of the element stiffness matrix were experienced. Namely, for many layered shells, extensive out-of-core matrix manipulations were required, including the inversion of the stress field matrix for the elimination of the stress coordinates in terms of the generalized nodal displacments. These out-of-core operations resulted in prohibitive computer time requirements, even for a linear problem. Therefore, alternate approaches were investigated, which included layer lumping schemes and the use of displacement-type formulations with the capability of incorporating an arbitrary shear stress variation across the shell thickness.

In Section II, a systematic development of initial stress formulations is provided, which is then specialized for the analysis of general toroidal shells in Section III. In Sections IV and V, the appropriate numerical formulations are described, which include the hybrid stress finite element formulation and a displacement-type finite element approach which is based on a piecewise parabolic shear stress variation across the tire thickness.

The Fortran extended version of the Fortran-Compass computer code is written for the CDC-6600 machine under the Scope 3.3 and Scope 3.4.3 operating systems.

SECTION II

BASIC FORMULATIONS FOR INITIAL STRESS PROBLEMS

For clarity in presentation, the ideas will first be developed here within the framework of flat plate theory.

Each material particle in the original configuration C_1 is identified, in general, by three curvilinear coordinates ξ_i (i = 1, 2, 3). The numerical values of ξ_i which define a particle in C_1 define the same particle in every subsequent configuration (also referred to as convected coordinates). To describe the motion of the body relative to C_1, a fixed rectangular cartesian coordinate system x_i (i = 1, 2, 3) in three-dimensional space is also established.

In general, C_N is defined to be the configuration of the body before the addition of the n-th increment of load; whereas, C_{N+1} is the configuration of the body after the addition of the n-th load increment. In configuration C_N, the states of stress, strain, and deformation are presumed to be known. During the process of the n-th load increment, configuration C_N is treated to be in a state of "initial stress". Incremental displacements due to the addition of the n-th load increment are measured from C_N. In the following, as a generic case, the movement of the body from the reference state C_N to the deformed state C_{N+1} through small but finite increments in stresses, displacements, and external loads is treated.

The position vector of a particle in C_N is denoted by $\bar{r}$ and that of the same particle in C_{N+1} is denoted by $\bar{R}$. If x_i are the cartesian coordinates of the point in C_N and $\bar{e}_i$ are cartesian bases, it follows

$$\bar{r} = x_i \bar{e}_i \tag{2.1}$$

The covariant base vectors tangent to ξ_i lines in configuration C_N are then given by

$$\bar{g}_k = \frac{\partial \bar{r}}{\partial \xi_k} = \frac{\partial x_i}{\partial \xi_k} \bar{e}_i \tag{2.2}$$

and the covariant and contravariant metric tensors and contravariant base vectors in C_N are given by

$$g_{rs} = \bar{g}_r \, \bar{g}_s$$

$$\bar{g}^k = g^{kr} \, \bar{g}_r$$

$$g^{rs} \, g_{st} = \delta^r_t \tag{2.3}$$

where δ^r_t is unity when $r = t$, and zero otherwise.

For a flat plate, ξ_1 and ξ_2 are taken as the reference surface coordinates and ξ_3 as the thickness coordinate.

In such a coordinate system, the vector field of incremental displacements from C_N to C_{N+1} is measured in the basis system of C_N as

$$\bar{u} = u_\alpha \, \bar{g}_\alpha + w \, \bar{g}^3 + \xi_3 \, \omega_\alpha \, \bar{g}^\alpha \tag{2.4}$$

which gives the usual representation of Green's strain tensor in C_N as

$$e_{\alpha\beta} = \frac{1}{2} (u_{\alpha,\beta} + u_{\beta,\alpha} + u_{\nu,\alpha} \, u_{\nu,\beta} + w_{,\alpha} \, w_{,\beta})$$

$$k_{\alpha\beta} = \frac{1}{2} (\omega_{\alpha,\beta} + \omega_{\beta,\alpha} + u_{\gamma,\alpha}\, \omega_{\gamma,\beta} + u_{\gamma,\beta}\, \omega_{\gamma,\alpha})$$

$$\gamma_{\alpha} = \omega_{\alpha} + w_{,\alpha} + \omega_{\gamma}\, u_{\gamma,\alpha} \tag{2.5}$$

where

$e_{\alpha\beta}$ = membrane strain,

$u_{\alpha\beta}$ = curvature strain, and

γ_{α} = shear strain, and

ω_{α} = total rotation.

In the reference state C_N, which is presumed to be known, let the initial stress resultants be defined by

$$n^{\circ}_{\alpha\beta} = \int_h \tau^{\circ}_{\alpha\beta}\, d\xi_3$$

$$m^{\circ}_{\alpha\beta} = \int_h \xi_3\, \tau^{\circ}_{\alpha\beta}\, d\xi_3$$

$$r^{\circ}_{\alpha} = \int_h \tau^{\circ}_{\alpha 3}\, d\xi_3 \tag{2.6}$$

where

$n^{\circ}_{\alpha\beta}$ = membrane stress resultant,

$m^{\circ}_{\alpha\beta}$ = couple stress resultant,

r°_{α} = shear stress resultant,

$\tau^{\circ}_{\alpha\beta}, \tau^{\circ}_{\alpha 3}$ = Piola-Kirchoff stresses, measured per unit area in C_N, and

h = plate thickness.

Let the initial surface loads and edge tractions in the current reference state C_N be given by

$$\bar{f}^{\circ} = (f^{\circ}_{\alpha}, p^{\circ})$$

$$\bar{n} = (n_{\alpha}, r)$$

$$\bar{m} = (m_{\alpha}) \tag{2.7}$$

One can then prescribe additional surface forces (f_{α}, p), additional edge tractions $(\hat{n}_{\alpha}, \hat{r}, \hat{m}_{\alpha})$ on a portion S_1 of the boundary, and additional displacements and rotations $(\hat{u}_{\alpha}, \hat{w}, \hat{\omega}_{\alpha})$ on a portion S_2 of the boundary. Let the corresponding increment in the Piola-Kirchoff resultants be represented by $n_{\alpha\beta}$, $m_{\alpha\beta}$, and r_{α}. The principle of virtual work then states,

$$\int_A [(n^{\circ}_{\alpha\beta} + n_{\alpha\beta})\delta e_{\alpha\beta} + (m^{\circ}_{\alpha\beta} + m_{\alpha\beta})\delta k_{\alpha\beta} + (r^{\circ}_{\alpha} + r_{\alpha})\delta\gamma_{\alpha} - (f^{\circ}_{\alpha} + f_{\alpha})\delta u_{\alpha}$$

$$- (p^{\circ} + p)\delta w]dA - \int_{S_1} [(n^{\circ}_{\alpha} + \hat{n}_{\alpha})\delta u_{\alpha} + (m^{\circ}_{\alpha} + \hat{m}_{\alpha})\delta\omega_{\alpha}$$

$$+ (r^{\circ}_{\alpha} + r_{\alpha})\delta w]dS = 0 \tag{2.8}$$

where the strains are given by Equations (2.5). In Equation (2.8), the area A and the boundary S_1 refer to the known current reference state C_N and δ denotes variations. Note that u_{α}, w, and ω_{α} vanish on S_2.

Equation (2.8) may be written as

$$\int_A [n_{\alpha\beta}\, \delta e_{\alpha\beta} + m_{\alpha\beta}\, \delta k_{\alpha\beta} + r_{\alpha}\, \delta\gamma_{\alpha} + \frac{n^{\circ}_{\alpha\beta}}{2}\, \delta(w_{,\alpha}\, w_{,\beta} + u_{\nu,\alpha}\, u_{\nu,\beta})$$

$$+ \frac{m^{\circ}_{\alpha\beta}}{2}\, \delta(u_{\gamma,\alpha}\, \omega_{\gamma,\beta} + u_{\gamma,\beta}\, \omega_{\gamma,\alpha}) + r^{\circ}_{\alpha}\, \delta(\omega_{\gamma}\, u_{\gamma,\alpha}) - f_{\alpha}\, \delta u_{\alpha}$$

$$- p\delta w]dA - \int_{S_1} [n_{\alpha}\, \delta u_{\alpha} + m_{\alpha}\, \delta\omega_{\alpha} + r\delta w]dS = - \int_A [\frac{n^{\circ}_{\alpha\beta}}{2}\, \delta(u_{\alpha,\beta}$$

$$+ u_{\beta,\alpha}) + \frac{m^\circ_{\alpha\beta}}{2}\,\delta(\omega_{\alpha,\beta} + \omega_{\beta,\alpha}) + r^\circ_\alpha\,\delta(\omega_\alpha + w_{,\alpha})$$

$$- f^\circ_\alpha\,\delta u_\alpha - p^\circ\,\delta w]dA + \int_{S_1} [n^\circ_\alpha\,\delta u_\alpha + m^\circ_\alpha\,\delta\omega_\alpha$$

$$+ r^\circ\,\delta w]dS \qquad (2.9)$$

It is now assumed that the known initial stress state ($n^\circ_{\alpha\beta}$, $m^\circ_{\alpha\beta}$, r°_α, f°_α, p°, n°_α, m°_α, r°) in C_N is in equilibrium prior to the addition of the incremental loads. Then, the right hand side of Equation (2.9) can be shown to be identically equal to zero. That is,

$$n^\circ_{\alpha\beta,\beta} + f^\circ_\alpha = 0$$

$$m^\circ_{\alpha\beta,\beta} - r^\circ_\alpha = 0$$

$$r^\circ_{\alpha,\alpha} + p^\circ = 0 \qquad (2.10)$$

and

$$n^\circ_\alpha = n^\circ_{\alpha\beta}\, n_\beta$$

$$m^\circ_\alpha = m^\circ_{\alpha\beta}\, n_\beta$$

$$r^\circ = r^\circ_\alpha\, n_\alpha \qquad (2.11)$$

where n_α ($\alpha = 1, 2$) are the components of the edge normal. However, due to the numerical incremental solution technique for solving the above

problem, the initial stress state C_N may not be in equilibrium. Following [8], it is possible to derive an equilibrium error check if the right-hand side terms in Equation (2.9) are retained.

Assuming now that the reference state C_N is one of equilibrium, Equation (2.9) can be shown to yield the equilibrium equations for the incremental resultants referred to the current known reference state C_N as follows,

$$
\begin{aligned}
& n_{\alpha\beta,\beta} + [(n_{\gamma\beta} + n^{\circ}_{\gamma\beta})u_{\alpha,\beta}]_{,\gamma} + [(m_{\gamma\beta} + m^{\circ}_{\gamma\beta})\omega_{\alpha,\beta}]_{,\gamma} \\
& \qquad + [(r_{\gamma} + r^{\circ}_{\gamma})\omega_{\alpha}]_{,\gamma} + f_{\alpha} = 0 \\
& m_{\alpha\beta,\beta} - r_{\alpha} + [(m_{\gamma\beta} + m^{\circ}_{\gamma\beta})u_{\alpha,\gamma}]_{,\beta} - [(r_{\gamma} + r^{\circ}_{\gamma})u_{\alpha,\gamma}] = 0 \\
& r_{\alpha,\alpha} + p + [(n_{\alpha\beta} + n^{\circ}_{\alpha\beta})w_{,\alpha}]_{,\beta} = 0
\end{aligned}
\qquad (2.12)
$$

and

$$
\begin{aligned}
& \hat{n}_{\alpha} = n_{\alpha\beta}\, n_{\beta} + (n_{\gamma\beta} + n^{\circ}_{\gamma\beta})u_{\alpha,\beta}\, n_{\gamma} + (m_{\gamma\beta} + m^{\circ}_{\gamma\beta})\omega_{\alpha,\beta}\, n_{\gamma} \\
& \qquad + (r_{\gamma} + r^{\circ}_{\gamma})\omega_{\alpha}\, n_{\gamma} \\
& \hat{m}_{\alpha} = m_{\alpha\beta} n_{\beta} + (m_{\gamma\beta} + m^{\circ}_{\gamma\beta})u_{\alpha,\gamma}\, n_{\beta} \\
& \hat{r} = r_{\alpha}\, n_{\alpha} + (n_{\alpha\beta} + n^{\circ}_{\alpha\beta})w_{,\alpha}\, n_{\beta} \qquad \text{on } S_1.
\end{aligned}
\qquad (2.13)
$$

on S_1.

The principle of virtual work as given by Equation (2.9) can now be generalized through the usual methods into a counterpart of the Hu-Washizu principle in linear elasticity [9].

Assuming that the elastic stress-strain relations are of the type

$$n_{\alpha\beta} = n^*_{\alpha\beta}(e_{\gamma\lambda}, k_{\gamma\lambda}, \gamma_\lambda)$$

$$m_{\alpha\beta} = m^*_{\alpha\beta}(e_{\gamma\lambda}, k_{\gamma\lambda}, \gamma_\lambda)$$

$$r_\alpha = r^*_\alpha(e_{\gamma\lambda}, k_{\gamma\lambda}, \gamma_\lambda) \tag{2.14}$$

or

$$e_{\alpha\beta} = e^*_{\alpha\beta}(n_{\gamma\lambda}, m_{\gamma\lambda}, r_\lambda)$$

$$k_{\alpha\beta} = k^*_{\alpha\beta}(n_{\gamma\lambda}, m_{\gamma\lambda}, r_\lambda)$$

$$\gamma_\alpha = \gamma^*_\alpha(n_{\gamma\lambda}, m_{\gamma\lambda}, r_\lambda) \tag{2.15}$$

one can define an elastic strain energy function

$$\delta E = n_{\alpha\beta}\,\delta e_{\alpha\beta} + m_{\alpha\beta}\,\delta k_{\alpha\beta} + r_\alpha\,\delta\gamma_\alpha \tag{2.16}$$

In addition, introduce the constraint conditions (2.15), and

$$u_\alpha = \hat{u}_\alpha$$

$$w = \hat{w}$$

$$\omega_\alpha = \hat{\omega}_\alpha \tag{2.17}$$

on S_2.

Then, one can formulate the generalized functional

$$\pi_g = \pi^*_g(e_{\alpha\beta}, k_{\alpha\beta}, \gamma_\alpha; u_\alpha, w, \omega_\alpha; n_{\alpha\beta}, m_{\alpha\beta}, r_\alpha; n_\alpha, m_\alpha, r) \tag{2.18}$$

where

$$\pi_g = \int_A \{E(e_{\alpha\beta}, k_{\alpha\beta}, \gamma_\alpha) + \frac{n^\circ_{\alpha\beta}}{2} (w_{,\alpha} w_{,\beta} + u_{\nu,\alpha} u_{\nu,\beta})$$

$$+ \frac{m^\circ_{\alpha\beta}}{2} (u_{\gamma,\alpha} \omega_{\gamma,\beta} + u_{\gamma,\beta} \omega_{\gamma,\alpha}) + r^\circ_\alpha(\omega_\gamma u_{\gamma,\alpha}) - f_\alpha u_\alpha$$

$$- pw - n_{\alpha\beta}[e_{\alpha\beta} - \frac{1}{2} (u_{\alpha,\beta} + u_{\beta,\alpha} + u_{\nu,\alpha} u_{\nu,\beta} + w_{,\alpha} w_{,\beta})$$

$$- m_{\alpha\beta}[k_{\alpha\beta} - \frac{1}{2} (\omega_{\alpha,\beta} + \omega_{\beta,\alpha} + u_{\gamma,\alpha} \omega_{\gamma,\beta} + u_{\gamma,\beta} \omega_{\gamma,\alpha})]$$

$$- r_\alpha[\gamma_\alpha - (\omega_\alpha + w_{,\alpha} + \omega_\gamma u_{\gamma,\alpha})]\}dA - \int_{S_1} (\hat{n}_\alpha u_\alpha + \hat{m}_\alpha \omega_\alpha$$

$$+ \hat{r}w)dS - \int_{S_2} [n_\alpha(u_\alpha - \hat{u}_\alpha) + m_\alpha(\omega_\alpha - \hat{\omega}_\alpha) + r(w - \hat{w})]dS$$

$$- \varepsilon^* \tag{2.19}$$

in which

$$\varepsilon^* = - \int_A [n^\circ_{\alpha\beta} u_{\alpha,\beta} + m^\circ_{\alpha\beta} \omega_{\alpha,\beta} + r^\circ_\alpha (\omega_\alpha + w_{,\alpha}) - f^\circ_\alpha u_\alpha - p^\circ w]dA$$

$$+ \int_{S_1} [n^\circ_\alpha u_\alpha + m^\circ_\alpha \omega_\alpha + r^\circ w]dS \tag{2.20}$$

In Equation (2.10), ε^* is the correction term to check the equilibrium of initial stress state in the reference state C_N. If the reference state is theoretically in equilibrium, then the variation of ε^* with respect to u_α, ω_α, and w is zero. It follows that the Euler equations corresponding to $\delta\pi_g = 0$ are:

(a) the equilibrium equations,

(b) strain-displacement relations,

(c) displacement boundary conditions,

(d) traction boundary conditions, and

(e) stress-strain relations:

$$n_{\alpha\beta} = \frac{\partial E}{\partial e_{\alpha\beta}}$$

$$m_{\alpha\beta} = \frac{\partial E}{\partial k_{\alpha\beta}}$$

$$r_{\alpha} = \frac{\partial E}{\partial \gamma_{\alpha}} \tag{2.21}$$

The assumption of Equation (2.16) is equivalent to the existence of a potential B such that

$$B(n_{\alpha\beta}, m_{\alpha\beta}, r_{\alpha}) = n_{\alpha\beta}\, e_{\alpha\beta} + m_{\alpha\beta}\, k_{\alpha\beta} + r_{\alpha}\, \gamma_{\alpha} - E \tag{2.22}$$

Now, using Equation (2.22) in Equation (2.19), one obtains the functional $\pi_R(n_{\alpha\beta}, m_{\alpha\beta}, r_{\alpha}; u_{\alpha}, w, \omega_{\alpha})$ such that

$$\begin{aligned}
\pi_R = \int_A \{ &- B(n_{\alpha\beta}, m_{\alpha\beta}, r_{\alpha}) + \frac{n^{\circ}_{\alpha\beta}}{2}(w_{,\alpha}\, w_{,\beta} + u_{\nu,\alpha}\, u_{\nu,\beta}) \\
&+ \frac{m^{\circ}_{\alpha\beta}}{2}(u_{\gamma,\alpha}\, \omega_{\gamma,\alpha} + u_{\gamma,\beta}\, \omega_{\gamma,\alpha}) + r^{\circ}_{\alpha}(\omega_{\gamma}\, u_{\gamma,\alpha}) - f_{\alpha}\, u_{\alpha} - pw \\
&+ \frac{n_{\alpha\beta}}{2}(u_{\alpha,\beta} + u_{\beta,\alpha} + u_{\nu,\alpha}\, u_{\nu,\beta} + w_{,\alpha}\, w_{,\beta}) + \frac{m_{\alpha\beta}}{2}(\omega_{\alpha,\beta} \\
&+ \omega_{\beta,\alpha} + u_{\gamma,\alpha}\, \omega_{\gamma,\beta} + u_{\gamma,\beta}\, \omega_{\gamma,\alpha}) + r_{\alpha}(\omega_{\alpha} + w_{,\alpha} + \omega_{\gamma}\, u_{\gamma,\alpha})\} dA \\
&- \int_{S_1} (\hat{n}_{\alpha}\, u_{\alpha} + \hat{m}_{\alpha}\, \omega_{\alpha} + \hat{r} w) dS - \int_{S_2} [r_{\alpha}(u_{\alpha} - \hat{u}_{\alpha}) + m_{\alpha}(\omega_{\alpha} - \hat{\omega}_{\alpha}) \\
&+ r(w - \hat{w})] dS - \varepsilon^{*}
\end{aligned} \tag{2.23}$$

Now, the variation of π_R with respect to the stress resultants ($n_{\alpha\beta}$, $m_{\alpha\beta}$, r_α) and the displacements (u_α, w; ω_α) yields the following:

(a) the equilibrium equations,

(b) displacement boundary conditions,

(c) traction boundary conditions, and

(d) displacement-gradient-resultant relations:

$$\frac{1}{2}(u_{\alpha,\beta} + u_{\beta,\alpha} + u_{\nu,\alpha} u_{\nu,\beta} + w_{,\alpha} w_{,\beta}) = \frac{\partial B}{\partial n_{\alpha\beta}}$$

$$\frac{1}{2}(\omega_{\alpha,\beta} + \omega_{\beta,\alpha} + u_{\gamma,\alpha} \omega_{\gamma,\beta} + u_{\gamma,\beta} \omega_{\gamma,\alpha}) = \frac{\partial B}{\partial m_{\alpha\beta}}$$

$$(\omega_\alpha + w_{,\alpha} + \omega_\gamma u_{\gamma,\alpha}) = \frac{\partial B}{\partial r_\alpha} \tag{2.24}$$

If in Equation (2.19) one assumes <u>a priori</u> that the strain-displacement relations (2.5) and the displacement boundary conditions (2.17) are satisfied, then one arrives at the functional

$$\pi_p = \int_A [E(u_\alpha, w, \omega_\alpha) + \frac{n^\circ_{\alpha\beta}}{2}(w_{,\alpha} w_{,\beta} + u_{\nu,\alpha} u_{\nu,\beta}) + \frac{m^\circ_{\alpha\beta}}{2}(u_{\gamma,\alpha} \omega_{\gamma,\beta}$$

$$+ u_{\nu,\beta} \omega_{\gamma,\alpha}) + r^\circ_\alpha(\omega_\gamma u_{\gamma,\alpha}) - f_\alpha u_\alpha - pw]dA - \int_{S_1} (\hat{n}_\alpha u_\alpha$$

$$+ \hat{m}_\alpha \omega_\alpha + \hat{r}w)dS - \varepsilon^* \tag{2.25}$$

which is the potential energy functional for the initially stressed plate.

At this stage, it is desirable to construct a theory which allows large deflections but requires the angles of rotations to be small compared to unity. For this degree of accuracy, it is necessary to retain

in the strain-displacement relations only those nonlinear terms which involve the product of the normal displacement gradients, and neglect all others. Thus,

$$e_{\alpha\beta} = \frac{1}{2}\,(u_{\alpha,\beta} + u_{\beta,\alpha} + w_{,\alpha}\,w_{,\beta})$$

$$k_{\alpha\beta} = \frac{1}{2}\,(\omega_{\alpha,\beta} + \omega_{\beta,\alpha})$$

$$\gamma_{\alpha} = \omega_{\alpha} + w_{,\alpha} \tag{2.26}$$

Then, the generalized functional π_g of Equation (2.19) takes the form

$$\begin{aligned}\pi_g = \int_A \{ & F(e_{\alpha\beta}, k_{\alpha\beta}, \gamma_{\alpha}) + \frac{n^{\circ}_{\alpha\beta}}{2}\,(w_{,\alpha}\,w_{,\beta}) - f_{\alpha}\,u_{\alpha} - pw - n_{\alpha\beta}[e_{\alpha\beta} \\ & - \frac{1}{2}\,(u_{\alpha,\beta} + u_{\beta,\alpha} + w_{,\alpha}\,w_{,\beta})] - m_{\alpha\beta}[k_{\alpha\beta} - \frac{1}{2}\,(\omega_{\alpha,\beta} + \omega_{\beta,\alpha})] \\ & - r_{\alpha}[\gamma_{\alpha} - (\omega_{\alpha} + w_{,\alpha})]\}dA - \int_{S_1} (\hat{n}_{\alpha}\,u_{\alpha} + \hat{m}_{\alpha}\,\omega_{\alpha} + \hat{r}w)dS \\ & - \int_{S_2} [n_{\alpha}(u_{\alpha} - \hat{u}_{\alpha}) + m_{\alpha}(\omega_{\alpha} - \hat{\omega}_{\alpha}) + r(w - \hat{w})]dS - \varepsilon^{*}\end{aligned} \tag{2.27}$$

The Euler equations corresponding to the above functional are the equilibrium equations

$$n_{\alpha\beta,\beta} + f_{\alpha} = 0$$

$$m_{\alpha\beta,\beta} - r_{\alpha} = 0$$

$$r_{\alpha,\alpha} + [(n_{\alpha\beta} + n^{\circ}_{\alpha\beta})w_{,\alpha}]_{,\beta} + p = 0 \tag{2.28}$$

with the boundary conditions

$$\hat{n}_\alpha = n_{\alpha\beta}\, n_\beta$$

$$\hat{m}_\alpha = m_{\alpha\beta}\, n_\beta$$

$$\hat{r} = r_\alpha\, n_\alpha + (n_{\alpha\beta} + n^\circ_{\alpha\beta}) w_{,\alpha}\, n_\beta \qquad (2.29)$$

on S_1, and

$$\hat{u}_\alpha = u_\alpha$$

$$\hat{w} = w$$

$$\hat{\omega}_\alpha = \omega_\alpha \qquad (2.30)$$

on S_2.

Furthermore, the strain-displacement relations as defined by (2.26) and the stress resultant-strain relations are:

$$n_{\alpha\beta} = \frac{\partial E}{\partial e_{\alpha\beta}}$$

$$m_{\alpha\beta} = \frac{\partial E}{\partial k_{\alpha\beta}}$$

$$r_\alpha = \frac{\partial E}{\partial \gamma_\alpha} \qquad (2.31)$$

Next, certain simplifications can be made in order to facilitate the above developments for numerical calculations. One can assume that each increment is such that the incremental displacements u_α, w, and rotations ω_α are of order $O(\varepsilon)$; whereas the initial stresses are of order $O(1)$.

Then, the incremental strain-displacement relations are

$$e_{\alpha\beta} = \frac{1}{2}\,(u_{\alpha,\beta} + u_{\beta,\alpha}) + O(\varepsilon^2)$$

$$k_{\alpha\beta} = \frac{1}{2}\,(\omega_{\alpha,\beta} + \omega_{\beta,\alpha})$$

$$\gamma_\alpha = \omega_\alpha + w_{,\alpha} \tag{2.32}$$

Similarly, for elastic materials,

$$n_{\alpha\beta} \simeq O(\varepsilon);\; n^\circ_{\alpha\beta} \simeq O(1)$$

$$m_{\alpha\beta} \simeq O(\varepsilon);\; m^\circ_{\alpha\beta} \simeq O(1)$$

$$r_\alpha \simeq O(\varepsilon);\; r^\circ_\alpha \simeq O(1) \tag{2.33}$$

Using Equations (2.32) and (2.33), the third equilibrium equation in (2.28) can be simplified as

$$r_{\alpha,\alpha} + (n^\circ_{\alpha\beta}\, w_{,\alpha})_{,\beta} + p = 0 \tag{2.34}$$

and the third traction condition of Equation (2.29) takes the form

$$\hat{r} = r_\alpha\, n_\alpha + n^\circ_{\alpha\beta}\, w_{,\alpha}\, n_\beta \tag{2.35}$$

With the above simplifications, the generalized functional π_g, the Reissner functional π_R, and the potential energy functional π_p take the forms

$$\pi_g = \int_A \{E(e_{\alpha\beta}, k_{\alpha\beta}, \gamma_\alpha) + \frac{n^\circ_{\alpha\beta}}{2} (w_{,\alpha} w_{,\beta}) - f_\alpha u_\alpha - pw - n_{\alpha\beta}[e_{\alpha\beta}$$

$$- \frac{1}{2} (u_{\alpha,\beta} + u_{\beta,\alpha})] - m_{\alpha\beta}[k_{\alpha\beta} - \frac{1}{2} (\omega_{\alpha,\beta} + \omega_{\beta,\alpha})] - r_\alpha[\gamma_\alpha$$

$$- (\omega_\alpha + w_{,\alpha})]\}dA - \int_{S_1} (\hat{n}_\alpha u_\alpha + \hat{m}_\alpha \omega_\alpha + \hat{r}w)dS$$

$$- \int_{S_2} [n_\alpha(u_\alpha - \hat{u}_\alpha) + m_\alpha(\omega_\alpha - \hat{\omega}_\alpha) + r(w - \hat{w})]dS - \varepsilon^* \qquad (2.36)$$

$$\pi_R = \int_A [- B(n_{\alpha\beta}, m_{\alpha\beta}, r_\alpha) + \frac{n^\circ_{\alpha\beta}}{2} (w_{,\alpha} w_{,\beta}) - f_\alpha u_\alpha - pw$$

$$+ \frac{n_{\alpha\beta}}{2} (u_{\alpha,\beta} + u_{\beta,\alpha}) + \frac{m_{\alpha\beta}}{2} (\omega_{\alpha,\beta} + \omega_{\beta,\alpha}) + r_\alpha(\omega_\alpha + w_{,\alpha})]dA$$

$$- \int_{S_1} (\hat{n}_\alpha u_\alpha + \hat{m}_\alpha \omega_\alpha + \hat{r}w)dS - \int_{S_2} [n_\alpha(u_\alpha - \hat{u}_\alpha) + m_\alpha(\omega_\alpha - \hat{\omega}_\alpha)$$

$$+ r(w - \hat{w})]dS - \varepsilon^* \qquad (2.37)$$

and

$$\pi_p = \int_A [E(u_\alpha, w, \omega_\alpha) + \frac{n^\circ_{\alpha\beta}}{2} w_{,\alpha} w_{,\beta} - f_\alpha u_\alpha - pw]dA - \int_{S_1} (\hat{n}_\alpha u_\alpha$$

$$+ \hat{m}_\alpha \omega_\alpha + \hat{r}w)dS - \varepsilon^* \qquad (2.38)$$

At this point, note that the functional of Equation (2.37) leads to a mixed numerical formulation while the functional of (2.38) leads to the displacement formulation of finite element analysis.

If in Equation (2.36) one assumes that the linearized incremental equilibrium equations, the boundary traction conditions, and the stress

resultant-strain equations are satisfied a priori, then the following functional can be constructed by utilizing Equation (2.22),

$$\pi_c = - \int_A B(n_{\alpha\beta}, m_{\alpha\beta}, r_\alpha)dA + \int_{S_2} (n_\alpha \hat{u}_\alpha + m_\alpha \hat{\omega}_\alpha + \hat{r}w)dS - \varepsilon^* \tag{2.39}$$

which is now suitable for the hybrid stress finite element formulation with the following modifications.

First, note that the functional ε^* is a constant with respect to variations in the resultants $n_{\alpha\beta}$, $m_{\alpha\beta}$, and r_α, and hence does not contribute any terms in the variational equation $\delta\pi_c = 0$. However, an equilibrium check on the initial stresses can be performed by retaining ε^* in Equation (2.39), as in the usual displacement formulation. If the condition

$$\begin{aligned} n_\alpha &= \hat{n}_\alpha \\ m_\alpha &= \hat{m}_\alpha \\ r &= \hat{r} \end{aligned} \tag{2.40}$$

on S_1 is satisfied a priori, then one can introduce it as a subsidiary condition and consider

$$\pi_c^* = - \int_A B(n_{\alpha\beta}, m_{\alpha\beta}, r_\alpha)dA + \int_{S_2} (n_\alpha \hat{u}_\alpha + m_\alpha \hat{\omega}_\alpha + r\hat{w})dS$$

$$+ \int_{S_1} [(\hat{n}_\alpha - n_\alpha)u_\alpha + (\hat{m}_\alpha - m_\alpha)\omega_\alpha + (\hat{r} - r)w]dS - \varepsilon^* \tag{2.41}$$

The variational equation $\delta\pi_c^* = 0$ then leads to the linearized strain-displacement boundary conditions on S_2 (2.30).

It is emphasized that one assumes *a priori* that the linearized equilibrium equations are satisfied in terms of the Piola-Kirchoff resultants, taken per unit area in C_N as

$$n_{\alpha\beta,\beta} + f_\alpha = 0$$

$$m_{\alpha\beta,\beta} - r_\alpha = 0$$

$$r_{\alpha,\alpha} + (n^\circ_{\alpha\beta}\, w_{,\alpha})_{,\beta} + p = 0 \tag{2.42}$$

Following Pian and Tong[4], one can modify the functional in Equation (2.41) for the hybrid stress formulation of the finite element model as

$$\pi_c^* = \sum_{n=1}^{M} \{ - \int_{A_n} B(n_{\alpha\beta}, m_{\alpha\beta}, r_\alpha)dA + \int_{\partial A_n} (n_\alpha u_\alpha + m_\alpha \omega_\alpha + rw)dS \}$$

$$- \varepsilon^* \tag{2.43}$$

where M is the number of elements, A_n is the area of the n-th element, and ∂A_n is the interelement boundary of the n-th element. In the above equation, u_α, ω_α, and w are the interelement boundary displacements and rotations, and the meaning of Lagrangian multipliers can be used to satisfy the interelement traction continuity requirement on the average. These interelement boundary displacements and rotations are prescribed such that they inherently satisfy the interelement compatibility conditions.

The numerical facilitation of Equations (2.38) and (2.43) for the stress analysis of pneumatic tires will be discussed in Section IV.

SECTION III

INITIAL STRESS FORMULATIONS FOR GENERAL TOROIDAL SHELLS

A. Geometrical Considerations

The linearized field equations in the presence of initial stresses are developed here for toroidal shells with arbitrary reference meridian profiles, following the basic ideas of Section II.

Let x_i be rectangular cartesian coordinates with the origin fixed in space, as indicated in Figure 1. The position vector of a point P is defined by

$$\bar{r} = x_i \, \bar{e}_i \tag{3.1}$$

where $\bar{e}_i$ are unit base vectors. Introducing the curvilinear coordinate system,

$$\begin{aligned} x_1 &= (r_0 + \zeta \cos \xi_2) \cos \xi_1 \\ x_2 &= (r_0 + \zeta \cos \xi_2) \sin \xi_1 \\ x_3 &= \zeta \sin \xi_2 \\ \zeta &= \zeta(\xi_2) \end{aligned} \tag{3.2}$$

the base vectors of the above reference surface become

$$\bar{a}_1 = (r_0 + \zeta \cos \xi_2)\bar{e}_\Theta$$

$$\bar{a}_2 = (\zeta \cos \xi_2)_{,2} \, \bar{e}_r + (\zeta \sin \xi_2)_{,2} \, \bar{e}_3$$

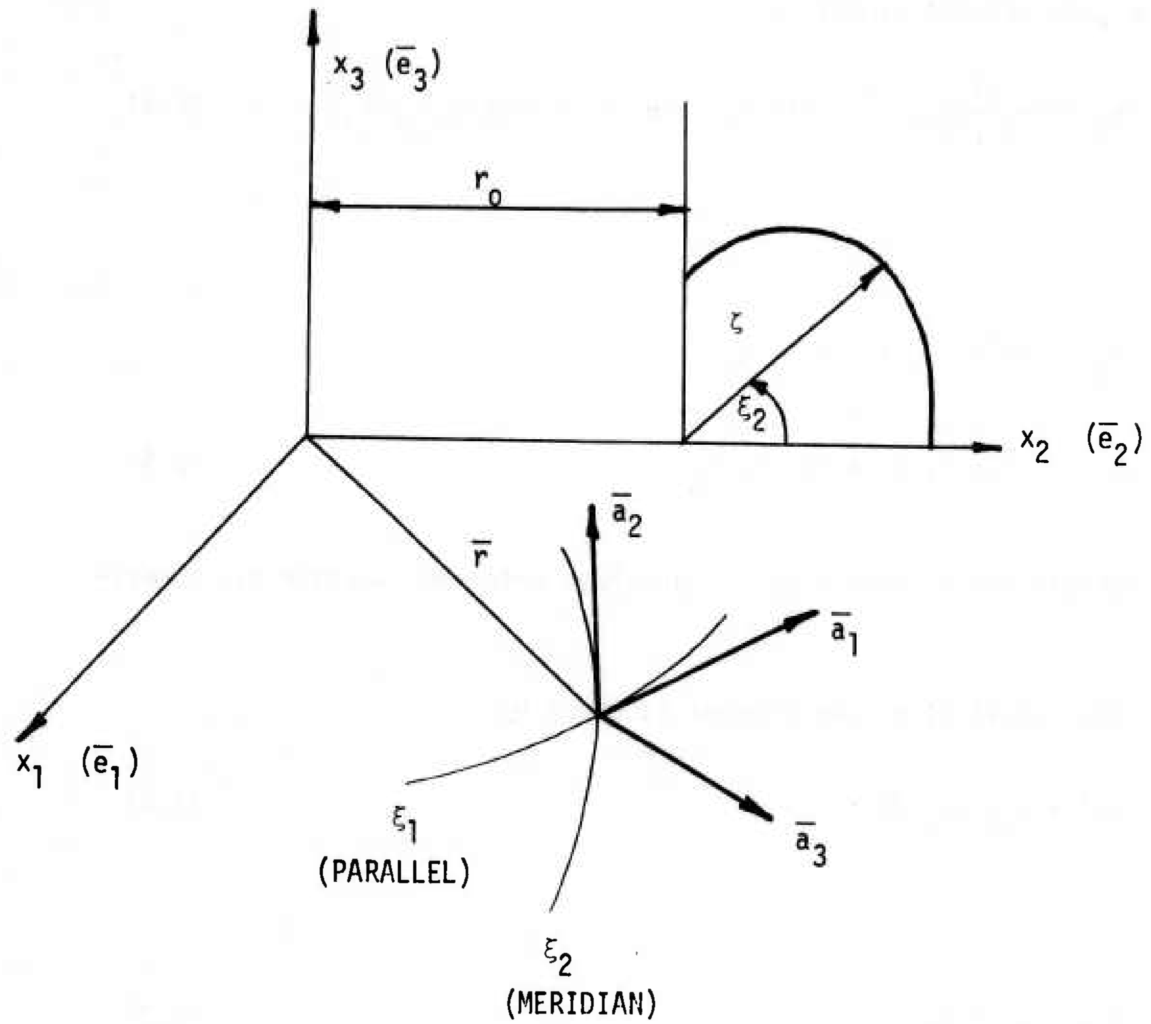


Figure 1. Reference Frames

and the unit outward normal is

$$\bar{a}_3 = \frac{1}{\sqrt{\zeta^2 + \zeta^2_{,2}}} \left[(\zeta \sin \xi_2)_{,2}\, \bar{e}_r - (\zeta \cos \xi_2)_{,2}\, \bar{e}_3\right] \tag{3.4}$$

where

$$\bar{e}_r = \cos \xi_1\, \bar{e}_1 + \sin \xi_1\, \bar{e}_2$$

$$\bar{e}_\Theta = - \sin \xi_1\, \bar{e}_1 + \cos \xi_1\, \bar{e}_2 \tag{3.5}$$

Next, certain basic properties of the above reference surface are summarized.

The length of a line element is given by

$$ds^2 = a_{\alpha\beta}\, d\xi_\alpha\, d\xi_\beta \tag{3.6}$$

where

$$a_{\alpha\beta} = \bar{a}_\alpha \cdot \bar{a}_\beta \tag{3.7}$$

Thus,

$$a_{11} = (r_0 + \zeta \cos \xi_2)^2$$

$$a_{22} = \zeta^2 + \zeta^2_{,2}$$

$$a_{12} = 0 \tag{3.8}$$

Therefore, ξ_1 and ξ_2 are the orthogonal curvilinear coordinates.

The second fundamental form of the reference surface is

$$d\bar{a}_3 \cdot d\bar{r} = b_{\alpha\beta}\, d\xi_\alpha\, d\xi_\beta \tag{3.9}$$

where the second metric tensor components are

$$b_{11} = -\,(r_0 + \zeta \cos \xi_2)\, \frac{(\zeta \sin \xi_2)_{,2}}{\sqrt{\zeta^2 + \zeta^2_{,2}}}$$

$$b_{22} = -\,(\zeta^2 + 2\,\zeta^2_{,2} - \zeta\zeta_{,2})/\sqrt{\zeta^2 + \zeta^2_{,2}}$$

$$b_{12} = 0 \tag{3.10}$$

Thus, the reference coordinate systems are those of the lines of curvature coordinates. The radii of curvatures are

$$\frac{1}{R_1} = -\,\frac{b_{11}}{a_{11}}$$

$$\frac{1}{R_2} = -\,\frac{b_{22}}{a_{22}} \tag{3.11}$$

The Gauss-Codazzi relations take the form

$$b_{\alpha\beta,\gamma} = b_{\alpha\gamma,\beta} \tag{3.12}$$

or

$$R_2 \cos \omega = \frac{dr}{d\omega} \tag{3.13}$$

where

$$\sin \omega = r/R_1 \tag{3.14}$$

The non-zero Crystoffel symbols, defined by

$$\bar{a}_{\alpha,\beta} = \Gamma^{\nu}_{\alpha\beta}\,\bar{a}_{\nu} + b_{\alpha\beta}\,\bar{a}_3 \tag{3.15}$$

are found to be

$$\Gamma^2_{11} = (r_0 + \zeta\cos\xi_2)(\zeta\cos\xi_2)_{,2}/(\zeta^2 + \zeta^2_{,2})$$

$$\Gamma^1_{12} = \Gamma^1_{21} = \frac{(\zeta\cos\xi_2)_{,2}}{r_0 + \zeta\cos\xi_2}$$

$$\Gamma^2_{22} = \frac{(\zeta\cos\xi_2)_{,2}\,(\zeta\cos\xi_2)_{,22} + (\zeta\sin\xi_2)_{,22}\,(\zeta\sin\xi_2)_{,2}}{\zeta^2 + \zeta^2_{,2}} \tag{3.16}$$

Covariant differentiation on the surface is defined in the usual manner,

$$A_{\alpha,\beta} = \frac{\partial A_\alpha}{\partial\xi_\beta} - \Gamma^{\lambda}_{\alpha\beta}\,A^{\lambda}$$

$$A^{\alpha}_{,\beta} = \frac{\partial A^\alpha}{\partial\xi_\beta} + \Gamma^{\alpha}_{\beta\lambda}\,A^{\lambda}$$

$$A_{\alpha\beta,\gamma} = \frac{\partial A_{\alpha\beta}}{\partial\xi_\gamma} - \Gamma^{\lambda}_{\alpha\gamma}\,A_{\lambda\beta} - \Gamma^{\lambda}_{\beta\lambda}\,A_{\alpha\lambda} \tag{3.17}$$

where

$$A^{\alpha} = a^{\alpha\beta}\,A_{\beta} \tag{3.18}$$

The position vector of a particle not on the reference surface is given by

$$\bar{r} = \bar{r}\,(\xi_1,\ \xi_2)\ \bar{e}_r + \xi_3\ \bar{a}_3 \tag{3.19}$$

where ξ_3 is the thickness coordinate.

The covariant base vectors of the shell space can now be written as

$$\bar{g}_\alpha = \bar{a}_\alpha + \xi_3 \frac{\partial \bar{a}_3}{\partial \xi_\alpha}$$

$$\bar{g}_3 = \bar{a}_3 \tag{3.20}$$

The metric tensor of the shell space becomes

$$g_{\alpha\beta} = \bar{g}_\alpha \cdot \bar{g}_\beta = a_{\alpha\beta} + 2\ \xi_3\ b_{\alpha\beta} + \xi_3^2\ b^\lambda_\beta\ b_{\alpha\lambda}$$

$$g_{\alpha 3} = 0$$

$$g_{33} = 1 \tag{3.21}$$

In the above lines of curvature coordinate system,

$$\bar{g}_1 = \bar{a}_1\ (1 + \frac{\xi_3}{R_1})$$

$$\bar{g}_2 = \bar{a}_2\ (1 + \frac{\xi_3}{R_2}) \tag{3.22}$$

Thus, if the thickness is small compared to the radii of curvatures, then it may be assumed that

$$\bar{g}_\alpha = \bar{a}_\alpha$$

$$g_{\alpha\beta} = a_{\alpha\beta}$$

$$\det(g_{\alpha\beta}) = \det(a_{\alpha\beta}) \equiv a \tag{3.23}$$

This approximation means that the Christoffel symbols for the space are equal to their value on the reference surface.

B. Field Equations

Consider now that the shell is deformed due to external loads so that each point undergoes a displacement

$$\bar{u} = u_\alpha \, \bar{a}^\alpha + w\bar{a}^3 + \xi_3 \, \omega_\alpha \, \bar{a}^\alpha \tag{3.24}$$

where u_α and w are the reference surface displacements and ω_α denotes the rotation of the normal to the reference surface.

Then, the Green's strain tensor, according to the approximation embodied in Equations (2.26), takes the form

$$e_{\alpha\beta} = \frac{1}{2} (u_{\alpha,\beta} + u_{\beta,\alpha} + 2 \, b_{\alpha\beta} \, w + w_{,\alpha} \, w_{,\beta})$$

$$k_{\alpha\beta} = \frac{1}{2} (\omega_{\alpha,\beta} + \omega_{\beta,\alpha}) + b^\nu_\alpha \, b_{\nu\beta} \, w + \frac{1}{2} (b^\nu_\alpha \, u_{\nu,\beta} + b^\nu_\beta \, u_{\nu,\alpha})$$

$$\gamma_\alpha = \frac{1}{2} (\omega_\alpha + w_{,\alpha} - b_{\alpha\nu} \, u^\nu) \tag{3.25}$$

The stress resultants are defined in the usual fashion,

$$n^{\alpha\beta} = \int_h \tau^{\alpha\beta} \, d\xi_3$$

$$m^{\alpha\beta} = \int_h \xi_3 \, \tau^{\alpha\beta} \, d\xi_3$$

$$r^\alpha = \int_h \tau^{\alpha\beta} \, d\xi_3 \tag{3.26}$$

Using Equations (3.25) and (3.26), the principle of virtual displacements yields the following equilibrium equations:

$$n^{\alpha\beta}_{,\beta} + b^{\alpha}_{\nu} m^{\nu\beta}_{,\beta} + b^{\alpha}_{\nu} r^{\nu} + f^{\alpha} = 0$$

$$m^{\alpha\beta}_{,\beta} - r^{\alpha} = 0$$

$$r^{\alpha}_{,\alpha} - n^{\alpha\beta} b_{\alpha\beta} - m^{\alpha\beta} b^{\nu}_{\alpha} b_{\nu\beta} + (n^{\gamma\beta} w_{,\beta})_{,\alpha} + p = 0 \tag{3.27}$$

The corresponding edge tractions take the forms:

$$n^{\beta} = (n^{\alpha\beta} + b^{\beta}_{\nu} m^{\alpha\nu})\nu_{\nu}$$

$$m^{\alpha} = m^{\alpha\beta} \nu_{\beta}$$

$$r = r^{\alpha} \nu_{\alpha} + n^{\alpha\beta} w_{,\alpha} \nu_{\beta} \tag{3.28}$$

Finally, within the framework of the initial stress formulation outlined in Section II, the linearized field equations analogous to Equations (2.26), (2.28), and (2.29) are

$$e_{\alpha\beta} = \frac{1}{2} (u_{\alpha,\beta} + u_{\beta,\alpha} + 2 b_{\alpha\beta} w)$$

$$k_{\alpha\beta} = \frac{1}{2} (\omega_{\alpha,\beta} + \omega_{\beta,\alpha}) + b^{\nu}_{\alpha} b_{\nu\beta} w + \frac{1}{2} (b^{\nu}_{\alpha} u_{\nu,\beta} + b^{\nu}_{\beta} u_{\nu,\alpha})$$

$$\gamma_{\alpha} = \frac{1}{2} (\omega_{\alpha} + w_{,\alpha} - b_{\alpha\nu} u^{\nu}) \tag{3.29}$$

$$n^{\alpha\beta}_{,\beta} + b^{\alpha}_{\nu}\, m^{\nu\beta}_{,\beta} + b^{\alpha}_{\nu}\, r^{\nu} + f^{\alpha} = 0$$

$$m^{\alpha\beta}_{,\beta} - r^{\alpha} = 0$$

$$r^{\alpha}_{,\alpha} - n^{\alpha\beta}\, b_{\alpha\beta} - m^{\alpha\beta}\, b^{\nu}_{\alpha}\, b_{\nu\beta} + (n^{\circ\alpha\beta}\, w_{,\beta})_{,\alpha} + p = 0 \tag{3.30}$$

and

$$n^{\beta} = (n^{\alpha\beta} + b^{\beta}_{\nu}\, m^{\alpha\nu})\nu_{\nu}$$

$$m^{\alpha} = m^{\alpha\beta}\, \nu_{\beta}$$

$$r = r^{\alpha}\, \nu_{\alpha} + n^{\circ\alpha\beta}\, w_{,\alpha}\, \nu_{\beta} \tag{3.31}$$

where $n^{\circ\alpha\beta}$ are the initial membrane stress resultants.

Note at this point that if one is to pursue the hybrid stress finite element formulation, then one must satisfy the equilibrium equations (3.30) identically, which may introduce a certain degree of numerical complexity when one deals with "deep" shell elements. Certain simplifications may arise, however, when the shell element is sufficiently shallow.

First of all, recall that, for surfaces, the only non-zero components of the Riemann-Christoffel tensor $R_{\alpha\beta\nu\lambda}$ are equal to $\pm R_{1212}$. Moreover [10],

$$R_{1212} = ka \tag{3.32}$$

where k is the gaussian curvature of the surface.

From the Gauss-Codazzi relations,

$$R_{1212} = b \quad ; \; b = \det(b_{\alpha\beta}) \tag{3.33}$$

so

$$k = b/a \tag{3.34}$$

Consequently, whenever k is small enough or zero (developable surfaces), the Riemann-Christoffel tensor can be taken equal to zero, and then the order of covariant differentiation on the surface may be interchanged. In this case, a stress function may be employed to satisfy the homogeneous equilibrium equations. Otherwise, recourse should be made to the static-geometric analogy [11].

For flat elements or for developable surfaces, one recovers the plate equations of Section II.

C. Inertial Loads Due to Rotation

The position vector of a generic point in the deformed state is

$$\bar{R} = (\bar{r} + \xi_3 \, \bar{a}_3) + (u^\alpha \, \bar{a}_\alpha + w\bar{a}_3 + \xi_3 \, \omega^\alpha \, \bar{a}_\alpha) \tag{3.35}$$

If the shell rotates with a constant angular velocity,

$$\bar{\Omega} = \Omega \, \bar{e}_3 \tag{3.36}$$

Then, the absolute velocity of a generic point is

$$\bar{v} = \bar{\Omega} \times \bar{R} + \frac{\partial \bar{R}}{\partial t} \tag{3.37}$$

and the acceleration is

$$\bar{a} = \bar{\Omega} \times (\bar{\Omega} \times \bar{R}) + 2\,\bar{\Omega} \times \frac{\partial \bar{R}}{\partial t} + \frac{\partial^2 \bar{R}}{\partial t^2} \tag{3.38}$$

For the static analysis,

$$\frac{\partial \bar{R}}{\partial t} = 0 \tag{3.39}$$

so that

$$\bar{a} = \Omega^2 \{\bar{e}_3 \times [\bar{e}_3 \times (\bar{r} + \xi_3\,\bar{a}_3) + \bar{e}_3 \times (u^\alpha\,\bar{a}_\alpha + w\bar{a}_3 + \xi_3\,\omega^\alpha\,\bar{a}_\alpha)]\} \tag{3.40}$$

Since $|\bar{r}| >> (u^\alpha;\, w;\, \xi_3)$, the final acceleration is

$$\bar{a} = \Omega^2 [\bar{e}_3 \times (\bar{e}_3 \times \bar{r})] \tag{3.41}$$

Using Equation (3.1), the above acceleration becomes

$$\bar{a} = -\,\Omega^2\, x_\alpha\, \bar{e}_\alpha \quad (\alpha = 1,\, 2) \tag{3.42}$$

and the corresponding inertia force is

$$\bar{f} = -\,\Omega^2\, x_\alpha\, \bar{e}_\alpha \int_{(h)} \gamma(\xi_3)\, d\xi_3 \tag{3.43}$$

where γ denotes the mass density distribution across the shell thickness.

SECTION IV

MATERIAL CHARACTERIZATION

A. One-Ply Systems

A unidirectional ply of cord and rubber is shown in Figure 2. The cord direction, denoted by 1, and the direction perpendicular to the cord, denoted by 2, form a set of directions referred to as the principal directions of the ply. The constitutive law of this cord-rubber ply composite may be characterized by five elastic constants:

E_1 = longitudinal Young's modulus,

E_2 = transverse Young's modulus,

ν_{12} = major Poisson ratio,

ν_{21} = secondary Poisson ratio, and

G = in-plane shear modulus.

Thus, the single-layered composite has the constitutive law

$$\varepsilon_1 = \frac{\sigma_1}{E_1} - \frac{\nu_{21}}{E_2}\,\sigma_2$$

$$\varepsilon_2 = \frac{\sigma_2}{E_2} - \frac{\nu_{12}}{E_1}\,\sigma_1$$

$$\varepsilon_{12} = \frac{\sigma_{12}}{2G} \tag{4.1}$$

If

$$E_1\,\nu_{21} = E_2\,\nu_{12} \tag{4.2}$$

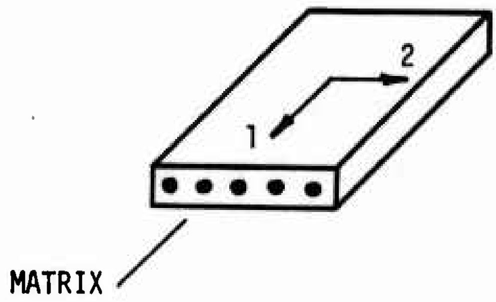


Figure 2. Unidirectional Ply

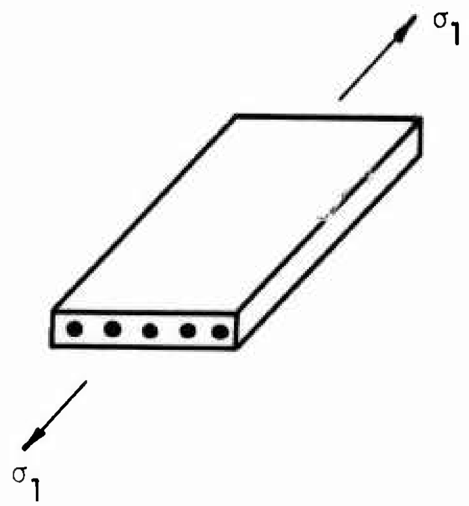


Figure 3. Tensile Test

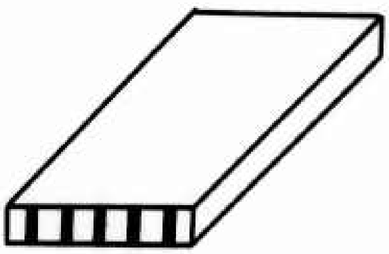


Figure 4. Simplified Model for Contraction

then the cord-ply system is referred to as a structurally orthotropic composite.

The rubber matrix and the reinforcing cord are assumed to be isotropic. Thus, for the rubber,

$$\varepsilon_1^R = \frac{1}{E_R}(\sigma_1^R - \nu_R\, \sigma_2^R)$$

$$\varepsilon_2^R = \frac{1}{E_R}(\sigma_2^R - \nu_R\, \sigma_1^R)$$

$$\varepsilon_{12}^R = \frac{\sigma_{12}^R}{2G_R} \tag{4.3}$$

and, for the cord,

$$\varepsilon_1^C = \frac{1}{E_C}(\sigma_1^C - \nu_C\, \sigma_2^C)$$

$$\varepsilon_2^C = \frac{1}{E_C}(\sigma_2^C - \nu_C\, \sigma_2^C)$$

$$\varepsilon_{12}^C = \frac{\sigma_{12}^C}{2G_C} \tag{4.4}$$

In the next section, the elastic constants of the composite will be determined in terms of the constituents. It will be assumed that the rubber matrix and the reinforcing cords are firmly bonded together.

B. Tensile Test

To determine the modulus of elasticity in the fiber direction, consider a test specimen as shown in Figure 3. The state of stress of this composite is

$$\sigma_1 \neq 0$$

$$\sigma_2 = 0$$

$$\sigma_{12} = 0 \tag{4.5}$$

Thus,

$$\varepsilon_1 = \frac{\sigma_1}{E_1}$$

$$\varepsilon_2 = -\nu_{12}\,\varepsilon_1$$

$$\varepsilon_{12} = 0 \tag{4.6}$$

Assuming perfect bond between the rubber matrix and reinforcing cords,

$$\varepsilon_1 = \varepsilon_1^R = \varepsilon_1^C \tag{4.7}$$

The condition of equilibrium in the 1 direction is

$$\int_A \sigma_1\,da = \int_{A_C} \sigma_1^C\,da + \int_{A_R} \sigma_1^R\,da \tag{4.8}$$

where

A = total cross-sectional area,

A_C = total cord area, and

A_R = cross-sectional area of the rubber matrix.

If one deals with mean values,

$$\sigma_1 = r\,\sigma_1^C + (1 - r)\sigma_1^R \tag{4.9}$$

where

$$r = \frac{A_C}{A} \tag{4.10}$$

The state of stress in the constituents is

$$\sigma_1^C \neq 0$$

$$\sigma_2^C = 0$$

$$\sigma_{12}^C = 0$$

$$\sigma_1^R \neq 0$$

$$\sigma_2^R = 0$$

$$\sigma_{12}^R = 0 \tag{4.11}$$

Thus,

$$\varepsilon_1 = \frac{\sigma_1^R}{E_R}$$

$$\varepsilon_2^R = -\,\nu_R\,\varepsilon_1^R$$

$$\varepsilon_{12}^{R} = 0$$

$$\varepsilon_{1}^{C} = \frac{\sigma_1^C}{E_C}$$

$$\varepsilon_{2}^{C} = - \nu_C \, \varepsilon_1^C$$

$$\varepsilon_{12}^{C} = 0 \tag{4.12}$$

Using Equations (4.6) and (4.12) in (4.9), one obtains

$$E_1 = (1 - r)E_R + r\,E_C \tag{4.13}$$

To determine the Poisson ratio ν_{12}, consider the simplified model shown by Figure 4. Based on this model, the total contraction is the sum of the contraction of the constituents,

$$\varepsilon_2 = r\,\varepsilon_2^C + (1 - r)\varepsilon_2^R \tag{4.14}$$

Putting Equations (4.6) and (4.12) in (4.14),

$$\nu_{12} = r\,\nu_C + (1 - r)\nu_R \tag{4.15}$$

C. <u>Strip Test</u>

The strip test in the 1 direction is a pure homogeneous strain, as in Figure 5,

$$\varepsilon_2 = 0 \tag{4.16}$$

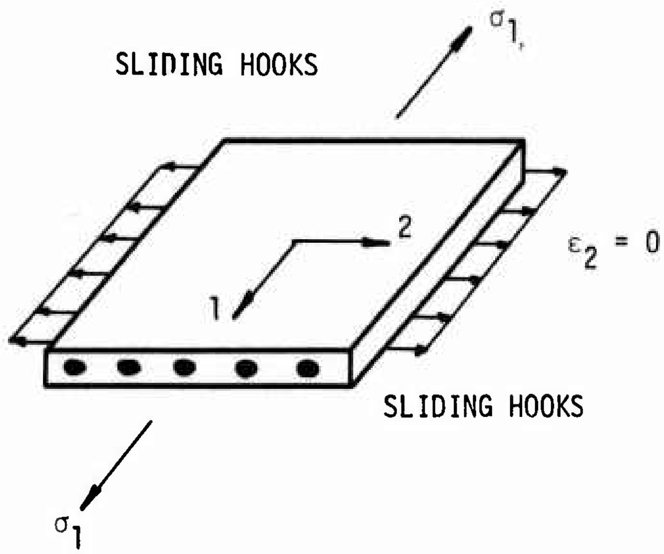


Figure 5. Strip Test

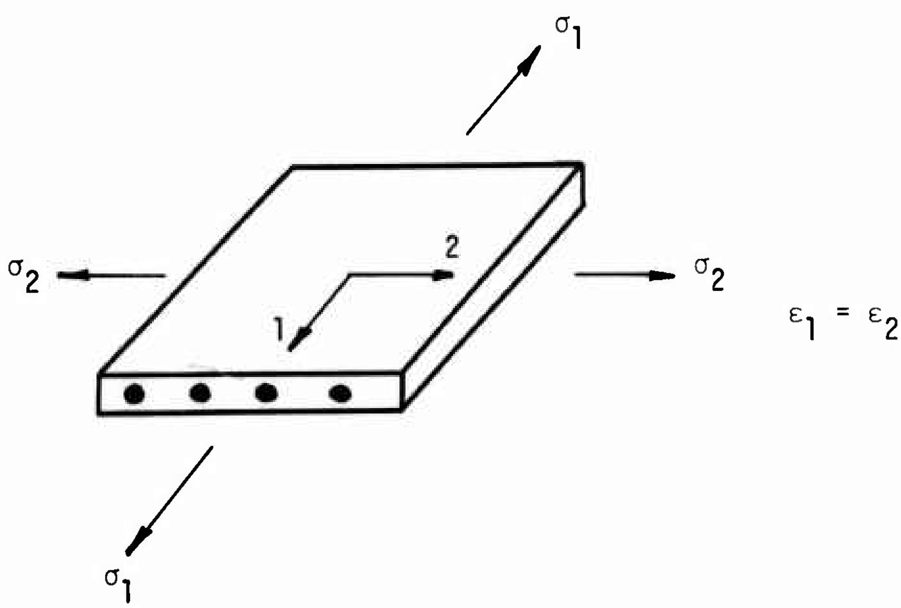


Figure 6. Sheet Test

Assuming perfect bonding,

$$\sigma_1 = (1 - r)\sigma_1^R + r\,\sigma_1^C \tag{4.17}$$

$$\varepsilon_1 = \varepsilon_1^R = \varepsilon_1^C \tag{4.18}$$

$$0 = (1 - r)\varepsilon_2^R + r\,\varepsilon_2^C \tag{4.19}$$

For the composite, one obtains

$$\sigma_2 = \frac{E_2}{E_1}\nu_{12}\,\sigma_1$$

$$\varepsilon_1 = \frac{\sigma_1}{E_1}(1 - \nu_{12}\,\nu_{21}) \tag{4.20}$$

Using Equation (4.18) and the averaged equilibrium condition in the 2 direction,

$$\sigma_2 = \sigma_2^R = \sigma_2^C \tag{4.21}$$

the constitutive relations for the constituents yield

$$\varepsilon_1 = \frac{1}{E_R}(\sigma_1^R - \nu_R\,\sigma_2)$$

$$\varepsilon_2^R = \frac{1}{E_R}(\sigma_2 - \nu_R\,\sigma_1^R) \tag{4.22}$$

$$\varepsilon_1 = \frac{1}{E_C}(\sigma_1^C - \nu_C\,\sigma_2)$$

$$\varepsilon_2^C = \frac{1}{E_C}(\sigma_2 - \nu_C\,\sigma_1^C) \tag{4.23}$$

From Equations (4.20), (4.22), and (4.23),

$$\sigma_1^R = E_R \, \varepsilon_1 + \nu_R \frac{E_2}{E_1} \nu_{12} \, \sigma_1$$

$$\sigma_1^C = E_C \, \varepsilon_1 + \nu_C \frac{E_2}{E_1} \nu_{12} \, \sigma_1 \tag{4.24}$$

The substitution of Equations (4.24) into (4.1) yields

$$E_2 \, \nu_{12} = E_1 \, \nu_{21} \tag{4.25}$$

Using Equations (4.20) and (4.24) in (4.22) and (4.23), one obtains

$$\varepsilon_2^R = \frac{1 - \nu_R^2}{E_R} \nu_{21} \, \sigma_1 - \nu_R \, \varepsilon_1$$

$$\varepsilon_2^C = \frac{1 - \nu_C^2}{E_C} \nu_{21} \, \sigma_1 - \nu_C \, \varepsilon_1 \tag{4.26}$$

Putting Equation (4.26) in (4.19), and employing (4.15), yields

$$\sigma_1 = \frac{\nu_{12}}{\nu_{21}} \frac{E_R \, E_C}{(1 - r)(1 - \nu_R^2)E_C + r(1 - \nu_C^2)E_R} \tag{4.27}$$

Comparing the above expression with Equation (4.20), one obtains

$$\frac{E_1}{1 - \nu_{12} \, \nu_{21}} = \frac{\nu_{12}}{\nu_{21}} \frac{E_R \, E_C}{(1 - r)(1 - \nu_R^2)E_C + r(1 - \nu_C^2)E_R} \tag{4.28}$$

With Equation (4.25), the above expression yields

$$\frac{E_2}{1 - \nu_{12}\,\nu_{21}} = \frac{E_R\,E_C}{(1 - r)(1 - \nu_R^2)E_C + r(1 - \nu_C^2)E_R} \tag{4.29}$$

or

$$E_2 = \frac{E_R\,E_C\,E_1}{E_1[(1 - r)(1 - \nu_R^2)E_C + r(1 - \nu_C^2)E_R] + E_R\,E_C\,\nu_{12}^2} \tag{4.30}$$

Eliminating E_1 and ν_{12} from Equation (4.30), one obtains the secondary Young's modulus in terms of the constituent properties,

$$E_2 = \frac{E_R\,E_C\,[(1 - r)E_R + r\,E_C]}{E_R\,E_C + r(1 - r)[(E_R - E_C)^2 - (E_R\,\nu_C - E_C\,\nu_R)^2]} \tag{4.31}$$

D. Sheet Test

The sheet test is a pure homogeneous strain with (Fig. 6)

$$\varepsilon_1 = \varepsilon_2 = \varepsilon \tag{4.32}$$

Again, employ the perfect bond condition and the equilibrium in the 1 and 2 directions,

$$\begin{aligned}
\varepsilon_1^R &= \varepsilon_1^C = \varepsilon \\
\varepsilon &= \varepsilon_2^R\,(1 - r) + r\,\varepsilon_2^C \\
\sigma_1 &= (1 - r)\sigma_1^R + r\,\sigma_1^C \\
\sigma_2^R &= \sigma_2^C = \sigma_2
\end{aligned} \tag{4.33}$$

For the composite, one obtains

$$\varepsilon = \frac{\sigma_1}{E_1} - \frac{\nu_{21}}{E_2}\,\sigma_2$$

$$\varepsilon = \frac{\sigma_2}{E_2} - \frac{\nu_{12}}{E_1}\,\sigma_1 \qquad (4.34)$$

or

$$\sigma_1 = \frac{E_1}{1 - \nu_{12}\,\nu_{21}}\,\varepsilon \qquad (4.35)$$

For the constituents,

$$\varepsilon_1^R = \frac{1}{E_R}\,(\sigma_1^R - \nu_R\,\sigma_2)$$

$$\varepsilon_2^R = \frac{1}{E_R}\,(\sigma_2 - \nu_R\,\sigma_1^R) \qquad (4.36)$$

and

$$\varepsilon_1^C = \frac{1}{E_C}\,(\sigma_1^C - \nu_C\,\sigma_2)$$

$$\varepsilon_2^C = \frac{1}{E_C}\,(\sigma_2 - \nu_C\,\sigma_1^C) \qquad (4.37)$$

Following the steps of the previous section, one obtains

$$E_2\,\nu_{12} = E_1\,\nu_{21} \qquad (4.38)$$

and

$$E_2 = \frac{E_R\, E_C[(1 - r)E_R + r\, E_C]}{E_R\, E_C + r(1 - r)[(E_R - E_C)^2 - (E_R\, \nu_C - E_C\, \nu_R)^2]} \tag{4.39}$$

E. <u>Pure Shear</u>

Consider the test illustrated in Figure 7. At the center of the specimen, the state of stress may be characterized by

$$\begin{aligned} \sigma_1 &= 0 \\ \sigma_2 &= 0 \\ \sigma_{12} &\neq 0 \end{aligned} \tag{4.40}$$

Using the simplified model, Figure 7, for the composite, the average shear may be written as

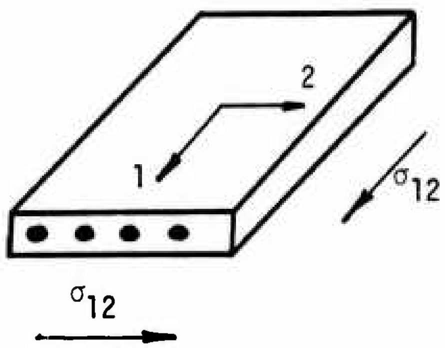
$$\varepsilon_{12} = (1 - r)\varepsilon_{12}^{R} + r\, \varepsilon_{12}^{C} \tag{4.41}$$

From static equilibrium,

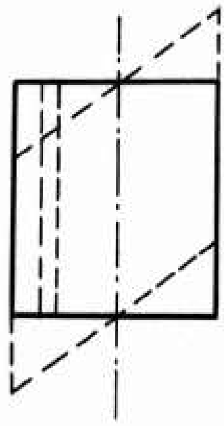
$$\sigma_{12}^{R} = \sigma_{12}^{C} = \sigma_{12} \tag{4.42}$$

Thus, the constitutive relations read,

$$\begin{aligned} \varepsilon_{12} &= \frac{\sigma_{12}}{G} \\ \varepsilon_{12}^{R} &= \frac{\sigma_{12}}{2G_R} \\ \varepsilon_{12}^{C} &= \frac{\sigma_{12}}{2G_C} \end{aligned} \tag{4.43}$$



(a) Pure Shear



(b) Simplified Model for Shear

Figure 7. Models for Pure Shear

Using Equation (4.43) in (4.41), one obtains

$$G = \frac{G_C \; G_R}{(1 - r)G_C + r \; G_R} \tag{4.44}$$

F. <u>Summary of In-Plane Test Results</u>

The elastic moduli for the one-ply composite are:

$$E_1 = E_C(1 - r)\lambda + r$$

$$\nu_{12} = r \; \nu_C + (1 - r)\nu_R$$

$$E_2 = E_R \frac{(1 - r)\lambda + r}{\lambda + r(1 - r)[(\lambda - 1)^2 - (\lambda\nu_C - \nu_R)^2]}$$

$$\nu_{21} = \frac{E_2}{E_1} \; \nu_{12}$$

$$G = G_C \frac{\omega}{(1 - r) + r\omega} \tag{4.45}$$

where

$$\lambda = E_R/E_C$$

$$\omega = G_R/G_C \tag{4.46}$$

Note that Equation (4.45) is not an <u>a priori</u> assumption, but a consequence of this modeling approach.

These formulas should be compared with the experimental results of the previously prescribed tests.

Thus, in the tensile tests,

(a) measure σ_1, ε_1, and ε_2, and

(b) calculate E_1 and ν_{12}.

In the strip or sheet tests,

(a) measure σ_1 and ε_1, and

(d) calculate $E_1/(1 - \nu_{12}\,\nu_{21}) = \sigma_1/\varepsilon_1$,

from which

$$\nu_{21} = \frac{\sigma_1 - E_1\,\varepsilon_1}{\nu_{12}} \tag{4.47}$$

Finally, using Equation (4.45),

$$E_2 = \frac{E_1\,\nu_{21}}{\nu_{12}} \tag{4.48}$$

G. <u>Material Properties in the Transverse Direction</u>

In the longitudinal direction,

$$\tau_{13} = r\,\tau_{13}^{C} + (1 - r)\tau_{13}^{R}$$

$$\tau_{13}^{C} = 2\,G_C\,\varepsilon_{13}$$

$$\tau_{13}^{R} = 2\,G_R\,\varepsilon_{13}$$

$$\tau_{13} = 2\,G_{13}\,\varepsilon_{13} \tag{4.49}$$

Thus,

$$G_{13} = r\,G_C + (1 - r)G_R \tag{4.50}$$

$$\varepsilon_{23} = (1 - r)\varepsilon_{23}^{R} + r\ \varepsilon_{23}^{C}$$

$$\tau_{23}^{R} = \tau_{23}^{C} = \tau_{23}$$

$$\varepsilon_{23}^{R} = \tau_{23}^{R}/2G_R$$

$$\varepsilon_{23} = \tau_{23}^{C}/2G_C$$

$$\varepsilon_{23} = \tau_{23}/2G_{23} \tag{4.51}$$

Thus,

$$G_{23} = \frac{G_R G_C}{(1 - r)G_C + rG_R} \tag{4.52}$$

H. Transformation to Local Frame

In general, the local coordinate system does not coincide with the coordinate system used to formulate the layer constitutive relations.

At each point on the reference surface, the cord orientation may be defined by the vector

$$\bar{c} = \cos\beta\ \frac{\bar{a}_1}{\sqrt{a_{11}}} + \sin\beta\ \frac{\bar{a}_2}{\sqrt{a_{22}}} \tag{4.53}$$

where the cord angle β is specified by the lift equation [7]

$$\cos\beta = (r_o + \zeta\cos\xi_2)\cos\gamma/r_b$$

$$r_b = \text{bead radius}$$

$$\gamma = \text{Green angle} \tag{4.54}$$

In the principal frame of reference ($\beta = 0$), the constitutive relations have the form

$$\varepsilon_1^* = \frac{\sigma_1^*}{E_1} - \frac{\nu_{21}}{E_2}\sigma_2^*$$

$$\varepsilon_2^* = \frac{\sigma_2^*}{E_2} - \frac{\nu_{12}}{E_1}\sigma_1^*$$

$$\gamma_{12}^* = \sigma_{12}^*/G_{12}$$

$$\gamma_{13}^* = \sigma_{13}^*/G_{13}$$

$$\gamma_{23}^* = \sigma_{23}^*/G_{23} \qquad (4.55)$$

The new elastic constants in the lines of curvature coordinate system are, [12]

$$\varepsilon_{11} = c_{11}\,\tau_{11} + c_{12}\,\tau_{22} + c_{13}\,\tau_{12}$$

$$\varepsilon_{22} = c_{12}\,\tau_{11} + c_{22}\,\tau_{22} + c_{23}\,\tau_{12}$$

$$\gamma_{12} = c_{13}\,\tau_{11} + c_{23}\,\tau_{22} + c_{33}\,\tau_{12}$$

$$\gamma_{13} = s_{11}\,\tau_{13} + s_{12}\,\tau_{23}$$

$$\gamma_{23} = s_{12}\,\tau_{13} + s_{22}\,\tau_{23} \qquad (4.56)$$

where

$$c_{11} = \frac{\cos^4 \beta}{E_1} + \left(\frac{1}{G_{12}} - \frac{2\nu_{12}}{E_1}\right) \sin^2 \beta \cos^2 \beta + \frac{\sin^4 \beta}{E_2}$$

$$c_{12} = \left(\frac{1}{E_1} + \frac{1}{E_2} + \frac{2\nu_{12}}{E_1} - \frac{1}{G_{12}}\right) \sin^2 \beta \cos^2 \beta - \frac{\nu_{12}}{E_1}$$

$$c_{13} = [2\left(\frac{\sin^2 \beta}{E_2} - \frac{\cos^2 \beta}{E_1}\right) + \left(\frac{1}{G_{12}} - \frac{2\nu_{12}}{E_1}\right) (\cos^2 \beta - \sin^2 \beta)]$$

$$\quad x \sin \beta \cos \beta$$

$$c_{22} = \frac{\sin^4 \beta}{E_1} + \left(\frac{1}{G_{12}} \frac{2\nu_{12}}{E_1}\right) \sin^2 \beta \cos^2 \beta + \frac{\cos^4 \beta}{E_2}$$

$$c_{23} = [2\left(\frac{\cos^2 \beta}{E_2} - \frac{\sin^2 \beta}{E_1}\right) - \left(\frac{1}{G_{12}} - \frac{2\nu_{12}}{E_1}\right) (\cos^2 \beta - \sin^2 \beta)]$$

$$\quad x \sin \beta \cos \beta$$

$$c_{33} = 4\left(\frac{1}{E_1} + \frac{1}{E_2} + \frac{2\nu_{12}}{E_1} - \frac{1}{G_{12}}\right) \sin^2 \beta \cos^2 \beta + \frac{1}{G_{12}}$$

$$s_{11} = \frac{\sin^2 \beta}{G_{23}} + \frac{\cos^2 \beta}{G_{13}}$$

$$s_{12} = \left(\frac{1}{G_{23}} - \frac{1}{G_{13}}\right) \sin \beta \cos \beta$$

$$s_{22} = \frac{\cos^2 \beta}{G_{23}} + \frac{\sin^2 \beta}{G_{13}} \qquad (4.57)$$

I. In-Plane Cord Angle Change

Consider a filament emanating from a material point P in the undeformed configuration, characterized by the vector $d\bar{R}$. After deformation, the same filament will be characterized by the vector $d\bar{r}$, Figure 8. If $\underset{\sim}{F}$ is the deformation gradient, then, [13]

$$d\bar{r} = \underset{\sim}{F}\, d\bar{R} \tag{4.58}$$

Let the displacement vector be $\bar{u}$, such that

$$\bar{u} = \bar{r} - \bar{R} \tag{4.59}$$

Thus,

$$d\bar{u} = d\bar{r} - d\bar{R} = (\underset{\sim}{F} - \underset{\sim}{I})d\bar{R} \tag{4.60}$$

where $\underset{\sim}{I}$ is the unit map.

The stretch is defined by

$$\lambda = \frac{ds}{dS} \tag{4.61}$$

where

$$dS = |d\bar{R}|$$

$$ds = |d\bar{r}| \tag{4.62}$$

Using Equation (4.58), one obtains the following expression for the stretch,

$$\lambda\bar{n} = \underset{\sim}{F}\, \bar{N}$$

$$\lambda^2 = N \cdot \underset{\sim}{F}^T \underset{\sim}{F}\bar{N} \tag{4.63}$$

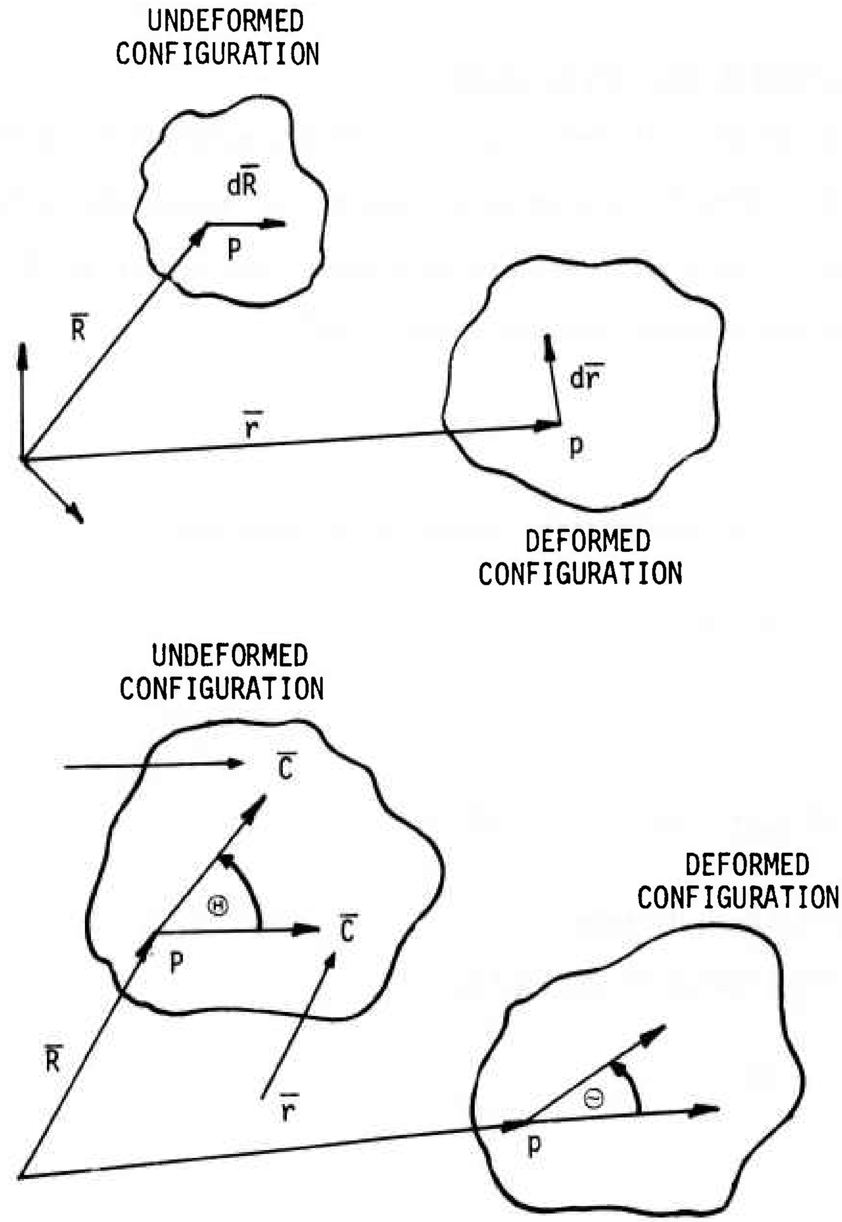


Figure 8. Angle Change Between Two Fibers

where

$$\bar{n} = \frac{d\bar{r}}{ds}$$

$$\bar{N} = \frac{d\bar{R}}{dS} \tag{4.64}$$

Define the displacement gradient as

$$d\bar{u} = \underset{\sim}{A}\, d\bar{R} \tag{4.65}$$

Comparing the above definition with Equation (4.60), one obtains

$$\underset{\sim}{A} = \underset{\sim}{F} - \underset{\sim}{I} \tag{4.66}$$

The classical strain tensor is defined by

$$\underset{\sim}{E} = \frac{1}{2} (\underset{\sim}{F}^T \underset{\sim}{F} - \underset{\sim}{I}) \tag{4.67}$$

or, using Equation (4.66),

$$\underset{\sim}{E} = \frac{1}{2} (\underset{\sim}{A} + \underset{\sim}{A}^T + \underset{\sim}{A}^T \underset{\sim}{A}) \tag{4.68}$$

In terms of the above strain tensor, the stretch defined by Equation (4.63) is

$$\lambda^2 = 2\, \bar{N} \cdot \underset{\sim}{E}\, \bar{N} + 1 \tag{4.69}$$

Note that if $\underset{\sim}{E} = \underset{\sim}{0}$, then the stretch is unity.

For deformations in the small strain domain, it is convenient to define the extension δ,

$$\delta = \lambda - 1 \tag{4.70}$$

Using Equation (4.70), one obtains the following expression for the extension,

$$\delta^2 + 2\,\delta = 2\,\overline{N} \cdot \underset{\sim}{E}\,\overline{N} \tag{4.71}$$

If $\delta^2 << 1$, then

$$\delta \simeq \overline{N} \cdot \underset{\sim}{E}\,\overline{N} \tag{4.72}$$

and it is spoken of as small extension.

In component form, Equation (4.72) reads

$$\delta = E_{\alpha\beta}\, N^{\alpha}\, N^{\beta} \tag{4.73}$$

where

$$\overline{N} = N^{\alpha}\, \overline{a}_{\alpha}$$

$$\underset{\sim}{E} = E_{\alpha\beta}\, (\overline{a}^{\alpha} \times \overline{a}^{\beta} \tag{4.74}$$

where $\overline{a}^{\alpha}$ and $\overline{a}_{\alpha}$ are contravariant and covariant base vectors, respectively.

Consider now two fibers in the undeformed configuration, defined by

$$\overline{C}_1 = C_1^{\alpha}\, \overline{a}_{\alpha}$$

$$\bar{C}_2 = C_2^{\alpha}\, \bar{a}_{\alpha} \tag{4.75}$$

as shown by Figure 8. The shear of these two fibers is defined by γ,

$$\gamma = \Theta - \theta \tag{4.76}$$

where

Θ = the angle between the two fibers in the undeformed configuration, and

θ = the angle between the same two fibers in the deformed configuration.

Using Equation (4.63), the corresponding deformed directions are

$$\bar{n}_1 = \frac{1}{\lambda_1} \underset{\sim}{F}\, \bar{C}_1$$

$$\bar{n}_2 = \frac{1}{\lambda_2} \underset{\sim}{F}\, \bar{C}_2 \tag{4.77}$$

Thus,

$$\cos\theta = \bar{n}_1 \cdot \bar{n}_2 = \frac{1}{\lambda_1\, \lambda_2} \bar{C}_2 \cdot \underset{\sim}{F}^T \underset{\sim}{F}\, \bar{C}_1 \tag{4.78}$$

Since

$$\underset{\sim}{F}^T \underset{\sim}{F} = 2\, \underset{\sim}{E} + \underset{\sim}{I} \tag{4.79}$$

then Equation (4.78) yields

$$\cos\theta = \frac{1}{\lambda_1\, \lambda_2} [2\, \bar{C}_2 \cdot \underset{\sim}{E}\, \bar{C}_1 + \cos\Theta] \tag{4.80}$$

where

$$\cos \Theta = \bar{C}_1 \cdot \bar{C}_2 \tag{4.81}$$

Using Equation (4.76),

$$\cos \tilde{\Theta} = \cos \Theta \cos \gamma + \sin \Theta \sin \gamma \tag{4.82}$$

and the component form of (4.80) becomes

$$\cos \Theta \cos \gamma + \sin \Theta \sin \gamma = \frac{1}{\lambda_1 \lambda_2} [2 E_{\alpha\beta} C_1^{\alpha} C_2^{\beta} + \cos \Theta] \tag{4.83}$$

Note that the usual ($\Theta = \pi/2$) shear is

$$\sin \gamma = \frac{1}{\lambda_1 \lambda_2} [2 E_{\alpha\beta} C_1^{\alpha} C_2^{\beta}] \tag{4.84}$$

For small extensions, Equation (4.83) yields

$$\cos \Theta \cos \gamma + \sin \Theta \sin \gamma = 2 E_{\alpha\beta} C_1^{\alpha} C_2^{\beta} + \cos \Theta \tag{4.85}$$

Knowing C_1^{α}, C_2^{β}, Θ, and the strain tensor $E_{\alpha\beta}$, Equation (4.85) is a transcendental equation for the shear γ. For small shear,

$$\cos \gamma \simeq 1$$

$$\sin \gamma \simeq \gamma \tag{4.86}$$

one obtains

$$\gamma = \frac{2 E_{\alpha\beta} C_1^{\alpha} C_2^{\beta}}{\sin \Theta} \tag{4.87}$$

Knowing the shear γ, the angle between the two fibers after deformation is

$$\Theta = \text{\textcircled{H}} - \frac{2\,E_{\alpha\beta}\,C_1^{\alpha}\,C_2^{\beta}}{\sin \text{\textcircled{H}}} \tag{4.88}$$

The above cord angle change is not yet incorporated into the computer code.

SECTION V

NUMERICAL FORMULATIONS

A. Discretization

Flat triangular elements are used with nodes being located on the reference surface. For the rotationally symmetric problem, the global coordinate system is that of the lines of curvature coordinates of the reference configuration.

Each of these triangles is defined by its vertex position vectors $\bar{r}_1$, $\bar{r}_2$, and $\bar{r}_3$.

Using the vertex position vectors, the following local base vectors are generated,

$$\bar{g}_1 = \bar{\zeta}_3/|\bar{\zeta}_3|$$

$$\bar{g}_2 = \bar{g}_3 \times \bar{g}_1$$

$$\bar{g}_3 = (\bar{\zeta}_3 \times \bar{\zeta}_1)/|\bar{\zeta}_3 \times \bar{\zeta}_1| \tag{5.1}$$

where

$$\bar{\zeta}_1 = \bar{r}_3 - \bar{r}_2$$

$$\bar{\zeta}_3 = \bar{r}_2 - \bar{r}_1 \tag{5.2}$$

The origin of this local coordinate system is taken to be the centroid of the triangle. The cartesian components of the local base vectors

are defined by

$$\bar{g}_i = g_{ij}\,\bar{e}_j \qquad (i,\ j = 1,\ 2,\ 3) \tag{5.3}$$

The area integrations are carried out by gaussian cubatures over T_2 simplexes [14]. The procedure is now outlined for an integral of the form

$$I = \int_A F(\eta_1,\ \eta_2) d\eta_1\, d\eta_2 \tag{5.4}$$

Let the in-plane vertex coordinates be denoted by $v_{\alpha 1}$ and $v_{\alpha 2}$ in a barycentric local coordinate system. In order to use gaussian cubature formulae for the numerical integration of Equation (5.4), each triangle is mapped onto a standard triangle (0,0; 1,0; 0,1) by the following simplex transformation:

$$z_1(\eta_1,\ \eta_2) = a_1\,\eta_1 + a_2\,\eta_2 + a_3$$

$$z_2(\eta_1,\ \eta_2) = b_1\,\eta_1 + b_2\,\eta_2 + b_3 \tag{5.5}$$

where the constants a_i and b_i are calculated from the transformation constraints:

$$z_1(v_{11},\ v_{12}) = 0$$

$$z_1(v_{21},\ v_{22}) = 1$$

$$z_1(v_{31},\ v_{32}) = 0 \tag{5.6}$$

and

$$z_2(v_{11}, v_{12}) = 0$$

$$z_2(v_{21}, v_{22}) = 0$$

$$z_3(v_{31}, v_{32}) = 1 \tag{5.7}$$

Then, using the inverse of Equation (5.5),

$$\eta_1(z_1, z_2) = [(z_1 - a_3)b_2 - (z_2 - b_3)b_1]/(a_1 b_2 - a_2 b_1)$$

$$\eta_2(z_1, z_2) = [(z_2 - b_3)a_1 - (z_1 - a_3)a_2]/(a_1 b_2 - a_2 b_1) \tag{5.8}$$

the integral defined by Equation (5.4) becomes

$$I = |T| \int_0^1 \int_0^{1-z} F[\eta_1(z_1, z_2); \eta_2(z_1, z_2)]dz_1\, dz_2 \tag{5.9}$$

where the Jacobian T is defined by

$$T = \frac{\partial(\eta_1, \eta_2)}{\partial(z_1, z_2)} = \det \begin{vmatrix} \frac{\partial\eta_1}{\partial z_1} & \frac{\partial\eta_1}{\partial z_2} \\ \frac{\partial\eta_2}{\partial z_1} & \frac{\partial\eta_2}{\partial z_2} \end{vmatrix} \tag{5.10}$$

which is calculated from Equation (5.8).

Next, the limits of integration in Equation (5.9) are changed to [-1, 1] by

$$z_1 = (1 - \xi)/2$$

$$z_2 = (1 - \eta)(1 + \xi)/4 \tag{5.11}$$

Thus,

$$I = \frac{|T|}{8} \int_{-1}^{1} \int_{-1}^{1} f(\xi, \eta)(1 + \xi)d\xi\, d\eta \tag{5.12}$$

where

$$f(\xi, \eta) = F\{\eta_1[z_1(\xi), z_2(\xi, \eta)], \eta_2[z_1(\xi), z_2(\xi, \eta)]\} \tag{5.13}$$

Equation (5.12) is now in the desired form to use tabulated gaussian weights and nodes. For

$$I = \frac{|T|}{8} \sum_{i=1}^{G} \sum_{j=1}^{G} C_{ij}\, f(\xi_i, \eta_j)$$

$$C_{ij} = A_i\, B_j \tag{5.14}$$

where the weights A_i and B_j and the nodes ξ_i and η_j are tabulated in [14] according to the following schedule:

$$\int_{-1}^{1} h(x)(1 + x)dx = \sum_{i=1}^{G} A_i\, h(x_i)$$

$$\int_{-1}^{1} h(x)dx = \sum_{i=1}^{G} B_i\, h(x_i) \tag{5.15}$$

An alternative form of Equation (5.15) is

$$I = \frac{|T|}{8} \sum_{\Omega=1}^{H} C_\Omega \, f(P_\Omega)$$

$$H = G \times G \tag{5.16}$$

where P_Ω is the gaussian node defined by the pair $(\xi_\Omega, \eta_\Omega)$.

Product integrations are performed as follows:

$$[I] = \int_{\Delta}\int_{\eta_1\eta_2} [F(\eta_1, \eta_2)]^T [D] [G(\eta_1, \eta_2)] d\eta_1 \, d\eta_2 \tag{5.17}$$

where the compliance matrix is assumed to be constant over the element. Thus, one way to perform the designated integration is to carry out the multiplication to obtain

$$[I] = \int_{\Delta}\int_{\eta_1\eta_2} [E(\eta_1, \eta_2)] d\eta_1 \, d\eta_2 \tag{5.18}$$

where

$$E(\eta_1, \eta_2) = F^T (\eta_1, \eta_2) D \, G(\eta_1, \eta_2) \tag{5.19}$$

Each term in the E matrix is a simple polynomial whose integral may easily be tabulated using the above procedure.

However, at this development stage, a more flexible approach is adopted which performs integration on matrix products in general. Thus,

$$[I] = \frac{|T|}{8} \sum_{i,j=1}^{G} C_{ij} \, [f(\bar{s}_i, \eta_j)] \, [D] \, [g(\bar{s}_i, \eta_j)] \tag{5.20}$$

B. The Hybrid Stress Finite Element Formulation

The hybrid stress functional for the problem under consideration has the form, from Equation (2.43),

$$\pi = \sum_{n=1}^{N} \frac{1}{2} \iint_{R_N} \{\int([\sigma(\eta_1, \eta_2; x_3)]^T [D(x_3)][\sigma(\eta_1, \eta_2; x_3)]$$

$$+ 2[\tau(\eta_1, \eta_2; x_3)]^T [D(x_3)][\tau(\eta_1, \eta_2; x_3)])dx_3\}d\eta_1 \, d\eta_2$$

$$- \oint_{R_N} [\overline{N}(s) \cdot \overline{u}(s) + \overline{M}(s) \cdot \overline{\phi}(s)]ds \qquad (5.21)$$

where

D = direct stress compliance,

S = shear stress compliance,

σ = column $(\sigma_{11}, \sigma_{22}, \sigma_{12})$ = direct stress,

τ = column $(\sigma_{13}, \sigma_{23})$ = shear stress,

$\overline{N}(s) = (N_v, N_t, A)$ = boundary moment resultant vector,

$\overline{M}(s) = (M_t, M_v)$ = boundary moment resultant vector,

$\overline{u}(s)$ = boundary displacement vector, and

$\overline{\phi}(s)$ = boundary rotation vector.

The stress field for each layer is chosen such that the layer equilibrium, interlayer equilibrium, and overall resultant equilibrium for an element are satisfied. However, stress continuity along the interelement boundary need not be satisfied. Accordingly, the solution of the equilibrium equations is obtained as the sum of the homogeneous and particular solution, such that the homogeneous state of stress satisfies

$$\sigma_{\alpha\beta,\beta} + \sigma_{\alpha 3,3} = 0$$

$$\sigma_{\alpha 3,\alpha} = 0 \tag{5.22}$$

throughout the thickness of the shell. Thus, for each layer i,

$$\sigma^{i}_{\alpha\beta,\beta} + \sigma^{i}_{\alpha 3,3} = 0$$

$$\sigma^{i}_{\alpha 3,\alpha} = 0 \qquad (i = 1, 2 \ldots L) \tag{5.23}$$

where L is the number of layers. In order to satisfy interlayer equilibrium, one must have

$$\sigma^{i}_{\alpha 3}(\eta_1, \eta_2; h_i) = \sigma^{i-1}_{\alpha 3}(\eta_1, \eta_2; h_i) \tag{5.24}$$

where h_i is the thickness coordinate of the interface.

The particular stress field is obtained in terms of the stress resultants which satisfy the incremental equilibrium equations

$$N^{k}_{\alpha\beta,\beta} + F^{k}_{\alpha} = 0$$

$$M^{k}_{\alpha\beta,\beta} - Q^{k}_{\alpha} = 0$$

$$Q^{k}_{\alpha,\alpha} + (N^{k-1}_{\alpha\gamma} \quad w^{k}_{,\gamma})_{,\alpha} + F^{k}_{3} = 0 \tag{5.25}$$

for the k-th step in an initial stress formulation, where F^{k}_{α} and F^{k}_{3} are the k-th load increments, and w^k is the current normal displacement. In the case of the rotating tire, the external loads F^{k}_{i} are replaced by the

associated inertial forces. After calculating the stress resultants $N^k_{\alpha\beta}$, $M^k_{\alpha\beta}$, and Q^k_α in terms of the incremental loads, the particular stress field is obtained in the usual fashion,

$$\sigma^k_{\alpha\beta} = \frac{N^k_{\alpha\beta}}{h} + \frac{12\ x_3}{h^3}\ M^k_{\alpha\beta}$$

$$\sigma^k_{\alpha 3} = \frac{3}{2h}\left[1 - 4\left(\frac{x_3}{h}\right)^2\right]Q^k_\alpha \tag{5.26}$$

where h is the shell thickness.

Thus, the total stress field for each layer will have the form

$$\sigma = \sigma_\beta(\eta_1,\ \eta_2;\ x_3)\beta + \sigma_\alpha(\eta_1,\ \eta_2;\ x_3)\alpha + \sigma_q(\eta_1,\ \eta_2;\ x_3)q$$

$$\tau = \tau_\beta(\eta_1,\ \eta_2;\ x_3)\beta + \tau_\alpha(\eta_1,\ \eta_2;\ x_3)\alpha + \sigma_q(\eta_1,\ \eta_2;\ x_3)q \tag{5.27}$$

where

$\sigma_\beta(\eta_1,\ \eta_2;\ x_3)\beta$ = homogeneous direct stress field ,

$\sigma_\alpha(\eta_1,\ \eta_2;\ x_3)\alpha$ = particular direct stress field due to the external incremental loads, and

$\sigma_q(\eta_1,\ \eta_2;\ x_3)q$ = particular direct stress field due to the presence of normal displacements in the third resultant equilibrium.

with identical definition for the shear stress distribution τ. Furthermore,

β = column $(\beta_1,\ \beta_2, \ldots \beta_{28})$ = undetermined stress coefficients,

α = column $(\alpha_1,\ \alpha_2, \ldots . \alpha_9)$ = external force vector obtained by linear interpolation of the nodal values, and

q = column $(q_1, q_2, \ldots q_{15})$ = nodal displacement vector for an element.

The vectors α and q are common to all layers, while the vector β for each layer is such that the interlayer equilibrium (5.24) is satisfied.

For each element, the edge resultants may be calculated from the stress field (5.27) to yield

$$\bar{N}(s) = [N_\nu^\beta(s)\beta + N_\nu^\alpha(s)\alpha]\bar{\nu} + [N_t^\beta(s) + N_t^\alpha(s)]\bar{t} + [Q^\beta(s)\beta + Q^\alpha(s)\alpha + Q^q(s)q]\bar{n}$$

$$\bar{M}(s) = [M_t^\beta(s)\beta + M_t^\alpha(s)\alpha + M_t^q(s)q]\bar{\nu} + [M_\nu^\beta(s)\beta + M_\nu^\alpha(s)\alpha + M_\nu^q(s)q]\bar{t} \tag{5.28}$$

where $\bar{\nu}$ and $\bar{t}$ are the outward normal and tangent vectors along the edge of an element and $\bar{n}$ is the outward normal to the plane of the element.

The displacement and rotation vectors along the boundary of an element are

$$\bar{u}(s) = (U_t(s)q)\bar{t} + (U_\nu(s)q)\bar{\nu} + (W(s)q)\bar{n}$$

$$\bar{\phi}(s) = (\phi_t(s)q)\bar{t} + (\phi_\nu(s)q)\bar{\nu} \tag{5.29}$$

where the vectors $U_t(s)$, $U_\nu(s)$, $W(s)$, $\phi_t(s)$ and $\phi_\nu(s)$ imply interpolation along the boundary of an element in terms of the nodal displacements q_1, $q_2, \ldots q_{15}$.

Using Equations (5.27), (5.28), and (5.29) in (5.1), one obtains the following hybrid stress functional suitable for numerical computation,

$$\pi(\beta, q) = \sum_{n=1}^{N} \{\tfrac{1}{2} [\beta^T C_{\beta\beta} \beta + 2 \beta^T P_\beta + 2 \beta^T \Delta C_{\beta q} q$$

$$+ 2 \Delta P_\alpha^T q + q^T \Delta C_{qq} q] - [\beta^T W_{\beta q} q + z_\alpha^T q + q^T \Delta W_{qq} q]\} \quad (5.30)$$

where the vector β refers to all the stress coefficients for an element, and where

$$C_{\beta\beta} = \sum_{i=1}^{L} (c_{\beta\beta})_i$$

$$P_\beta = \sum_{i=1}^{L} (p_\beta)_i$$

$$\Delta C_{\beta q} = \sum_{i=1}^{L} (\Delta c_{\beta q})_i$$

$$\Delta P_\alpha = \sum_{i=1}^{L} (\Delta p_\alpha)_i$$

$$\Delta C_{qq} = \sum_{i=1}^{L} (\Delta c_{qq})_i$$

$$W_{\beta q} = \sum_{i=1}^{L} (w_{\beta q})_i$$

$$z_\alpha = W_{\alpha q}^T \alpha$$

with

$$c_{\beta\beta} = \int_\Delta \int_{h_{i+1}}^{h_i} [\sigma_\beta^T D\sigma_\beta + 2 \tau_\beta^T S\tau_\beta] dx_3 \, d\eta_1 \, d\eta_2$$

$$c_{\beta\alpha} = \int_{\Delta} \int_{h_{i+1}}^{h_i} [\sigma_\beta^T D\sigma_\alpha + 2\, \tau_\beta^T S\tau_\alpha]dx_3\, d\eta_1\, d\eta_2$$

$$\Delta c_{\beta q} = \int_{\Delta} \int_{h_{i+1}}^{h_i} [\sigma_\beta^T D\sigma_q + 2\, \tau_\beta^T S\tau_q]dx_3\, d\eta_1\, d\eta_2$$

$$\Delta c_{qq} = \int_{\Delta} \int_{h_{i+1}}^{h_i} [\sigma_q^T D\sigma_q + 2\, \tau_q^T S\tau_q]dx_3\, d\eta_1\, d\eta_2$$

$$\Delta c_{\alpha q} = \int_{\Delta} \int_{h_{i+1}}^{h_i} [\sigma_\alpha^T D\sigma_q + 2\, \tau_\alpha^T S\tau_q]dx_3\, d\eta_1\, d\eta_2$$

$$P_\beta = c_{\beta\alpha}\, \alpha$$

$$\Delta P_\alpha = \Delta^T c_{\alpha q}\, \alpha$$

$$w_{\beta q} = \oint\{[N_\nu^\beta(s)]^T U_\nu(s) + [N_t(s)]^T U_t(s) + [M_t^\beta(s)]^T \phi_\nu(s)$$
$$+ [M_\nu^\beta(s)]^T \phi_t(s) + [Q^\beta(s)]^T W(s)\}ds$$

$$W_{\alpha q} = \oint \{[N_\nu^\alpha(s)]^T U_\nu(s) + [N_t^\alpha(s)]^T U_t(s) + [M_t^\alpha(s)]^T \phi_\nu(s)$$
$$+ [M_\nu^\alpha(s)]^T \phi_t(s) + [Q^\alpha(s)]^T W(s)\}ds$$

$$\Delta W_{qq} = \oint \{[M_t^q(s)]^T \phi_\nu(s) + [M_\nu^q(s)]^T \phi_t(s) + [Q^q(s)]^T W(s)\}ds \quad (5.31)$$

For clarity in presentation, the following nomenclature is introduced:

$C_{\beta\beta}$ = element flexibility matrix,

P_{β} = element flexibility vector,

$\Delta C_{\beta q}$ = homogeneous incremental element flexibility matrix,

ΔP_{α} = incremental element flexibility vector,

ΔC_{qq} = particular incremental element flexibility matrix,

$W_{\beta q}$ = hybrid element load matrix,

Z_{α} = hybrid element load vector, and

ΔW_{qq} = incremental hybrid element load matrix.

The symbol M in Equations (5.31) implies merging procedures according to the interlayer equilibrium condition (5.24).

Since the stress coefficients β are independent for each element, the variation of π with respect to β yields for each element,

$$C_{\beta\beta}\, \beta + P_{\beta} + \Delta C_{\beta q}\, q - W_{\beta q} = 0 \tag{5.32}$$

from which

$$\beta = - C_{\beta\beta}^{-1} [P_{\beta} + (\Delta C_{\beta q} - W_{\beta q})q] \tag{5.33}$$

Next, substituting Equation (5.33) into (5.30), and taking the variation with respect to q, one obtains

$$\sum_{n=1}^{N} (kr - p)_n = 0 \tag{5.34}$$

where k and p are the element stiffness and load matrices, defined by

$$k = k_1 + k_2$$

$$p = p_1 + p_2 \tag{5.35}$$

where

$$k_1 = - W_{\beta q}^T C_{\beta\beta}^{-1} W_{\beta q}$$

$$k_2 = - \Delta C_{\beta q}^T C_{\beta\beta}^{-1} \Delta C_{\beta q} + (\Delta C_{\beta q}^T C_{\beta\beta}^{-1} W_{\beta q}) + (\Delta C_{\beta q}^T C_{\beta\beta}^{-1} W_{\beta q})^T$$

$$+ \Delta C_{qq} + \frac{1}{2} (\Delta W_{qq} + \Delta W_{qq}^T)$$

$$p_1 = z_\alpha - W_{\beta q}^T C_{\beta\beta}^{-1} P_\beta$$

$$p_2 = - \Delta P_\alpha + \Delta C_{\beta q}^T C_{\beta\beta}^{-1} P_\beta \qquad (5.36)$$

where

k_1 = linear hybrid stress element stiffness matrix,

k_2 = incremental hybrid stress element stiffness matrix,

p_1 = linear hybrid stress element load vector, and

p_2 = incremental hybrid stress element load vector.

From this point, the analysis follows the usual steps of the displacement method. Thus,

$$Kr = g \qquad (5.37)$$

where

$$K = \sum_{n=1}^{N} (k)_n = \text{structure stiffness matrix}$$

$$g = \sum_{n=1}^{N} (p)_n = \text{structure load vector}$$

$$r = \text{unrestrained generalized displacements} \qquad (5.38)$$

Finally, Equation (5.37) is solved for the unknown displacements r, which are used to update the geometrical configuration and to calculate the initial stress resultants exhibited in the resultant equilibrium equations.

B. Displacement Formulation with an Assumed Piecewise Parabolic Shear Stress Distribution

The linearized field equations of Section II are first summarized here for clarity in presentation.

(a) strain-displacement equations:

$$e_{\alpha\beta} = \frac{1}{2} (u_{\alpha,\beta} + u_{\beta,\alpha})$$

$$k_{\alpha\beta} = \frac{1}{2} (\omega_{\alpha,\beta} + \omega_{\beta,\alpha})$$

$$\gamma_{\alpha} = \omega_{\alpha} + w_{,\alpha} \tag{5.39}$$

(b) equilibrium equations:

$$n_{\alpha\beta,\beta} + f_{\alpha} = 0$$

$$m_{\alpha\beta,\beta} - r_{\alpha} = 0$$

$$r_{\alpha,\alpha} + (n^{\circ}_{\alpha\beta} w_{,\alpha})_{,\beta} + p = 0 \tag{5.40}$$

(c) boundary conditions:

$$u_{\alpha} = \hat{u}_{\alpha}$$

$$\omega_{\alpha} = \hat{\omega}_{\alpha}$$

$$w = \hat{w} \tag{5.41}$$

on S_2, and

$$\hat{n}_\alpha = n_{\alpha\beta}\,\nu_\beta$$

$$\hat{m}_\alpha = m_{\alpha\beta}\,\nu_\beta$$

$$\hat{r} = r_\alpha\,\nu_\alpha + n^\circ_{\alpha\beta}\,w_{,\alpha}\,\nu_\beta \tag{5.42}$$

on S_1.

(d) strain-stress relations:

$$e_{\alpha\beta} = \frac{\partial B}{\partial n_{\alpha\beta}}\,(n_{\alpha\beta}, m_{\alpha\beta}, r_\alpha)$$

$$k_{\alpha\beta} = \frac{\partial B}{\partial m_{\alpha\beta}}\,(n_{\alpha\beta}, m_{\alpha\beta}, r_\alpha)$$

$$\gamma_\alpha = \frac{\partial B}{\partial r_\alpha}\,(n_{\alpha\beta}, m_{\alpha\beta}, r_\alpha) \tag{5.43}$$

The Reissner functional for the above field takes the form

$$\begin{aligned}\pi_R = \int_A \{&- B(n_{\alpha\beta}, m_{\alpha\beta}, r_\alpha) + \frac{n^\circ_{\alpha\beta}}{2}\,w_{,\alpha}\,w_{,\beta} - f_\alpha\,u_\alpha - pw \\ &+ \frac{n_{\alpha\beta}}{2}\,(u_{\alpha,\beta} + u_{\beta,\alpha}) + \frac{m_{\alpha\beta}}{2}\,(\omega_{\alpha,\beta} + \omega_{\beta,\alpha}) + r_\alpha(\omega_\alpha + w_{,\alpha})\}dA \\ &- \int_{S_1} (\hat{n}_\alpha\,u_\alpha + \hat{m}_\alpha\,\omega_\alpha + \hat{r}w)dS - \int_{S_2} [n_\alpha(u_\alpha - \hat{u}_\alpha) \\ &+ m_\alpha(\omega_\alpha - \hat{\omega}_\alpha) + r(w - \hat{w})]dS\end{aligned} \tag{5.44}$$

The variational equation $\delta\pi_R = 0$ yields

(a) equilibrium equations,

(b) displacement boundary conditions,

(c) traction boundary conditions, and

(d) displacement gradient-resultant relations.

This variational principle will now be utilized to relate the reference surface strains to the layer stresses $\tau^k_{\alpha\beta}$ and $\tau^k_{\alpha 3}$ where k denotes the layer number. First, in order to cope with an arbitrary shear stress distribution across the thickness, a piecewise parabolic shear stress field is introduced in the following form:

$$\tau^k_{\alpha 3} = S^k(z) r_\alpha \tag{5.45}$$

and the direct stresses will be represented by

$$\tau^k_{\alpha\beta} = N^k(z) n_{\alpha\beta} + M^k(z) m_{\alpha\beta} \tag{5.46}$$

where k indicates the k-th layer, and the functions $M^k(z)$, $N^k(z)$, and $S^k(z)$ will be determined later in Section V.C.1.

For the purposes of the present discussion, it is sufficient to note that $S^k(z)$ is a second-degree polynomial in the thickness (z) coordinate.

In each layer, the local strain-stress relations are (see Figure 9 for cord angle description)

$$\varepsilon^k_{11} = c^k_{11}\,\tau^k_{11} + c^k_{12}\,\tau^k_{22} + c^k_{13}\,\tau^k_{12}$$

$$\varepsilon^k_{22} = c^k_{12}\,\tau^k_{11} + c^k_{22}\,\tau^k_{22} + c^k_{23}\,\tau^k_{12}$$

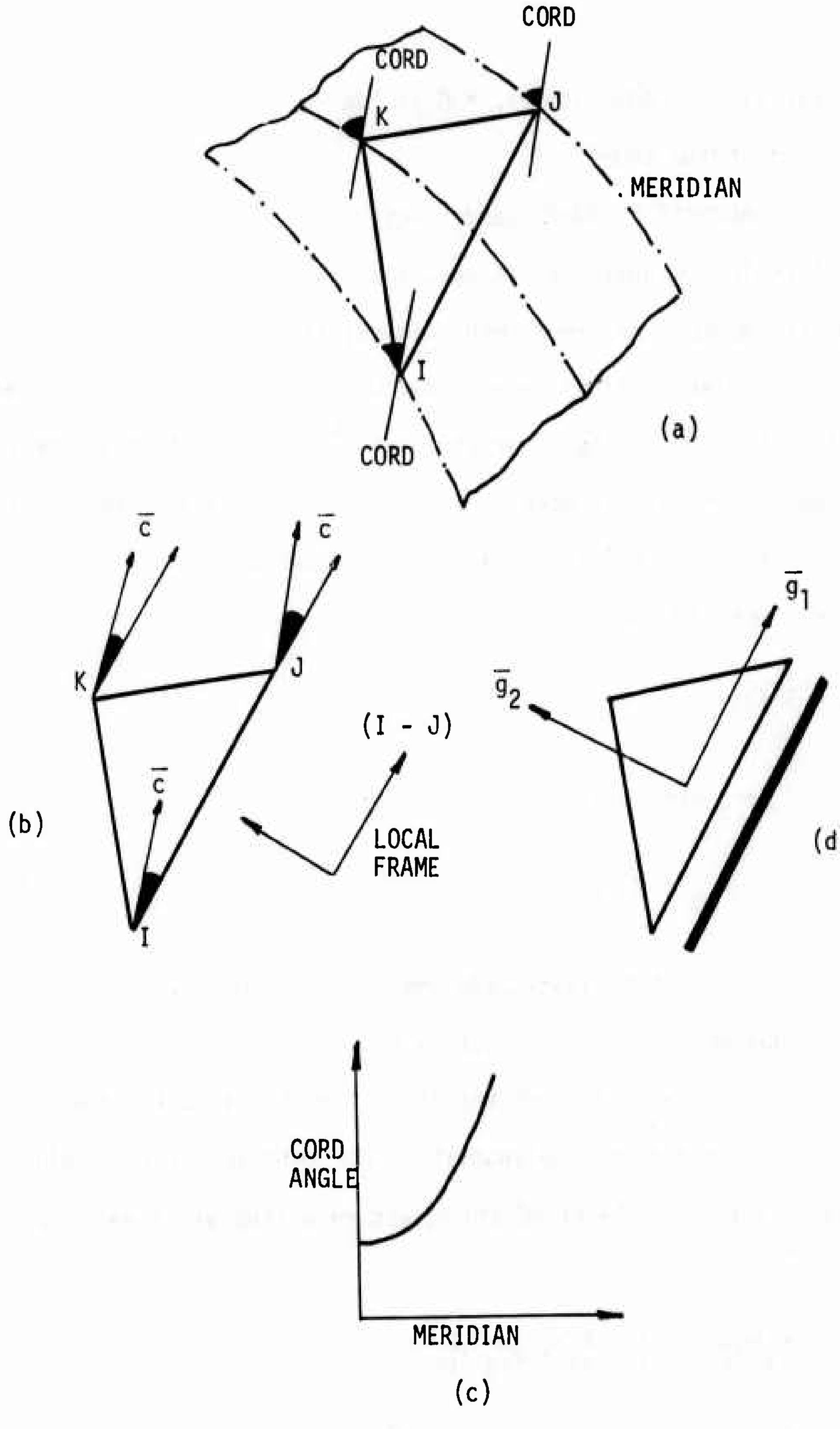


Figure 9. Cord Angle Description

$$\gamma_{12}^{k} = c_{13}^{k}\ \tau_{11}^{k} + c_{23}^{k}\ \tau_{22}^{k} + c_{33}^{k}\ \tau_{12}^{k}$$

$$\gamma_{13}^{k} = s_{11}^{k}\ \tau_{13}^{k} + s_{12}^{k}\ \tau_{23}^{k}$$

$$\gamma_{23}^{k} = s_{12}^{k}\ \tau_{13}^{k} + s_{22}^{k}\ \tau_{23}^{k} \qquad (5.47)$$

where the constants c_{ij}^{k} and s_{ij}^{k} are given by Equations (4.57) for the layer under consideration.

In short,

$$\varepsilon_{\alpha\beta}^{k} = C_{\alpha\beta\gamma\delta}^{k}\ \tau_{\gamma\delta}^{k}$$

$$\gamma_{\alpha}^{k} = D_{\alpha\beta}^{k}\ \tau_{\beta 3}^{k} \qquad (5.48)$$

The complementary energy density for the k-th layer is

$$B^{k} = C_{\alpha\beta\gamma\delta}^{k}\ \tau_{\alpha\beta}^{k}\ \tau_{\gamma\delta}^{k} + D_{\alpha\beta}^{k}\ \tau_{\alpha 3}^{k}\ \tau_{\beta 3}^{k} \qquad (5.49)$$

Using Equations (5.45) and (5.46) in (5.49), one obtains the complementary energy density in Equation (5.44) as follows:

$$B(n_{\alpha\beta},\ m_{\alpha\beta},\ r_{\alpha}) = \sum_{k=1}^{L} \int_{c_k}^{c_{k+1}} B^{k}(n_{\alpha\beta},\ m_{\alpha\beta},\ r_{\alpha};\ z)dz \qquad (5.50)$$

It follows that the variationally consistent reference surface strain-resultant relations are those of Equation (5.43). The inversion of these relations yields the resultant-strain relations in the form:

$$n_{\alpha\beta} = A_{\alpha\beta\gamma\delta}\ \ \varepsilon_{\gamma\delta} + B_{\alpha\beta\gamma\delta}\ k_{\gamma\delta}$$

$$m_{\alpha\beta} = B_{\alpha\beta\gamma\delta} \quad \varepsilon_{\gamma\delta} + C_{\alpha\beta\gamma\delta}\, k_{\gamma\delta}$$

$$r_\alpha = S_{\alpha\beta}\, \gamma_\beta \tag{5.51}$$

At this stage, recourse is made to the potential energy functional,

$$\pi_p = \int_A [E(u_\alpha, w, \omega_\alpha) + \frac{n^\circ_{\alpha\beta}}{2} w_{,\alpha} w_{,\beta} - f_\alpha u_\beta - pw]dA$$

$$- \int_{S_1} (\hat{n}_\alpha u_\alpha + \hat{m}_\alpha \omega_\alpha + \hat{r}w)dS \tag{5.52}$$

and thus, the analysis will be based on the displacement variational principle.

1. <u>Piecewise Parabolic Shear Stress Distribution</u>

For each layer, the components of the stress tensor are expressed in terms of the stress resultants as follows:

$$\tau^k_{\alpha\beta} = \frac{1}{A} \frac{d^2 \Theta^k}{dz^2} (z) n_{\alpha\beta} + \frac{1}{B} \frac{d\, \Theta^k}{dz} (z) m_{\alpha\beta}$$

$$\tau^k_{\alpha 3} = - \frac{1}{B} \Theta^k (z) r_\alpha \tag{5.53}$$

where k denotes the layer number, Figure 10, and the constants A and B, and the functions Θ^k (z) are established such that

$$n_{\alpha\beta} = \sum_{k=1}^{L} \int_{c_k}^{c_{k+1}} \tau^k_{\alpha\beta}(\eta_1, \eta_2; z)dz$$

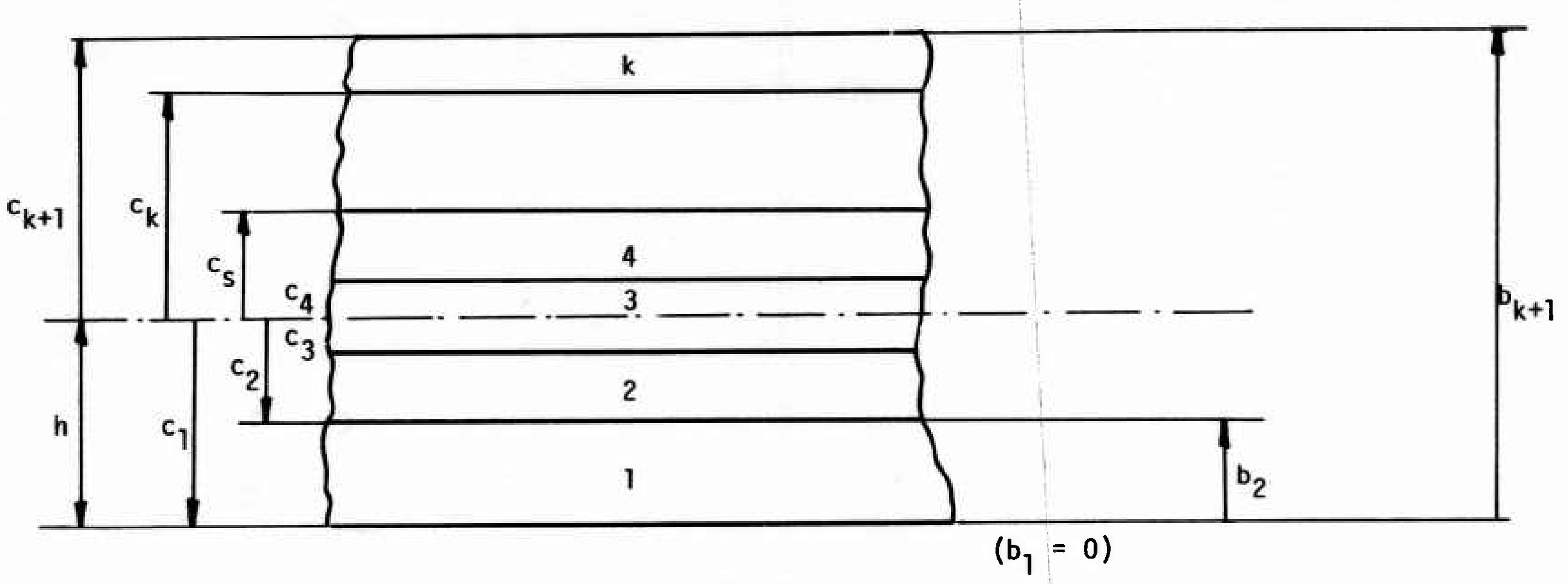


Figure 10. Geometrical Nomenclature for Piecewise Parabolic Shear Stress Distribution

$$m_{\alpha\beta} = \sum_{k=1}^{L} \int_{c_k}^{c_{k+1}} z\, \tau^k_{\alpha\beta}(\eta_1, \eta_2; z)dz$$

$$r_\alpha = \sum_{k=1}^{L} \int_{c_k}^{c_{k+1}} \tau^k_{\alpha 3}(\eta_1, \eta_2; z)dz \qquad (5.54)$$

where c_k and c_{k+1} denote the k-th layer surface positions as measured from the reference surface, Figure 10.

For each layer, the usual three-dimensional equilibrium equations are to be satisfied,

$$\tau^k_{\alpha\beta,\beta} + \tau^k_{\alpha 3,z} + f^k_\alpha = 0$$

$$\tau^k_{\alpha 3,\alpha} + (\tau^{\circ k}_{\alpha\beta}\, w_{,\beta})_{,\alpha} + \tau^k_{33,z} = 0 \qquad (5.55)$$

The overall equilibrium requires that the shear stresses are continuous across the boundaries of adjacent layers,

$$\tau^k_{\alpha 3}(c_{k+1}) = \tau^{k+1}_{\alpha 3}(c_{k+1}) \qquad [k = 1, 2, \ldots (L-1)] \qquad (5.56)$$

On the inner and outer surfaces, the shear stresses are zero,

$$\tau^1_{\alpha 3}(c_1) = 0$$

$$\tau^L_{\alpha 3}(c_{L+1}) = 0 \qquad (5.57)$$

Equations (5.54) require that

$$A = \sum_{k=1}^{L} \int_{c_k}^{c_{k+1}} \frac{d^2\,\Theta^k}{dz^2}(z)dz \qquad (5.58)$$

$$0 = \sum_{k=1}^{L} \int_{c_k}^{c_{k+1}} \frac{d\Theta^k}{dz}(z)dz \tag{5.59}$$

$$0 = \sum_{k=1}^{L} \int_{c_k}^{c_{k+1}} z \frac{d^2\Theta^k}{dz^2}(z)dz \tag{5.60}$$

$$B = \sum_{k=1}^{L} \int_{c_k}^{c_{k+1}} z \frac{d\Theta^k}{dz}(z)dz \tag{5.61}$$

$$B = -\sum_{k=1}^{L} \int_{c_k}^{c_{k+1}} \Theta^k(z)dz \tag{5.62}$$

Now, Equations (5.56) and (5.57) imply that Equation (5.59) is identically satisfied. It also follows that Equation (5.61) is identical to that of Equation (5.62).

Equation (5.60) essentially defines the reference surface location, Figure 10.

If the shear stress distribution across the shell thickness is approximated by a piecewise parabolic distribution, then $\Theta^k(z)$ has the form

$$\Theta^k(z) = s_k(z^2 - c_k^2) + S_k \tag{5.63}$$

where s_k and S_k are characteristics of the layer under consideration. Using the above form,

$$A = \sum_{k=1}^{L} 2\, s_k(c_{k+1} - c_k)$$

$$B = \sum_{k=1}^{L} \frac{2}{3} s_k (c_{k+1}^3 - c_k^3)$$

$$\sum_{k=1}^{L} s_k (c_{k+1}^2 - c_k^2) = 0 \tag{5.64}$$

From Figure 10,

$$c_k = b_k - h \tag{5.65}$$

so that the third of Equations (5.64) takes the form

$$h = \frac{1}{2} \sum_{k=1}^{L} s_k (b_{k+1}^2 - b_k^2) / \sum_{k=1}^{L} s_k (b_{k+1} - b_k) \tag{5.66}$$

From Equations (5.56) and (5.57), one obtains

$$S_1 = 0$$

$$S_k = S_{k-1} + s_{k-1}(c_k^2 - c_{k-1}^2) \qquad (k = 2, 3, \ldots L) \tag{5.67}$$

The stress tensor components for the layers are

$$\tau_{\alpha\beta}^k = \frac{2s_k}{A} n_{\alpha\beta} + \frac{2s_k}{B} z \, m_{\alpha\beta}$$

$$\tau_{\alpha 3}^k = -\frac{1}{B} [s_k(z^2 - c_k^2) + S_k] r_\alpha \tag{5.68}$$

Note that f_α^k in Equation (5.55) can be shown to be

$$f_\alpha^k = \frac{2s_k}{A} f_\alpha \tag{5.69}$$

and

$$p = \tau_{33}^{L}(c_{L+1}) - \tau_{33}^{1}(c_1) \tag{5.70}$$

The transverse normal stress from the second equations of (5.55) and (5.68) is found to be

$$\tau_{33}^{k}(z) = \frac{1}{B}\{s_k[\frac{1}{3}(z^3 - c_k^3) - c_k^2(z - c_k)] + S_k(z - c_k) + R_k\}r_{\alpha,\alpha} \tag{5.71}$$

in the absence of the initial stress term, where R_k is chosen such that the normal transverse stress is continuous across the boundaries of adjacent layers.

Using Equations (5.68), a variationally consistent set of reference strain-resultant relations are now derived according to the outline of Section V.C.

For compactness, the following notation is introduced:

$$\tau_1^k = \tau_{11}^k \qquad \varepsilon_1^k = \varepsilon_{11}^k$$

$$\tau_2^k = \tau_{22}^k \qquad \varepsilon_2^k = \varepsilon_{22}^k$$

$$\tau_3^k = \tau_{12}^k \qquad \varepsilon_3^k = \gamma_{12}^k$$

$$\tau_4^k = \tau_{13}^k \qquad \varepsilon_4^k = \gamma_{13}^k$$

$$\tau_5^k = \tau_{23}^k \qquad \varepsilon_5^k = \gamma_{23}^k$$

$$n_1 = n_{11} \qquad \varepsilon_1 = \varepsilon_{11}$$

$$n_2 = n_{22} \qquad \varepsilon_2 = \varepsilon_{22}$$

$$n_3 = n_{12} \qquad \varepsilon_3 = 2\,\varepsilon_{12}$$

$$n_4 = m_{11} \qquad \varepsilon_4 = k_{11}$$

$$n_5 = m_{22} \qquad \varepsilon_5 = k_{22}$$

$$n_6 = m_{12} \qquad \varepsilon_6 = 2\,k_{12}$$

$$n_7 = r_1 \qquad \varepsilon_7 = \gamma_{12}$$

$$n_8 = r_2 \qquad \varepsilon_8 = \gamma_{13} \tag{5.72}$$

Thus, the layer strain-stress relations (5.47) may be written as

$$\varepsilon_i^k = D_{ij}^k \tau_j^k \qquad (i, j = 1, 2, \ldots, 5) \tag{5.73}$$

The layer stress components are related to the resultants by

$$\tau_i^k = N_{ij}^k (z) n_j \qquad (i = 1, 2, \ldots, 5;\ j = 1, 2, \ldots, 8) \tag{5.74}$$

where the matrix $N_{ij}^k(z)$ is given in Appendix A.

The complementary energy (5.50) can now be written as

$$B = \frac{1}{2} \{ \sum_{k=1}^{L} D_{ij}^k [\int_{c_k}^{c_{k+1}} N_{ir}^k (z) N_{js}^k (z) dz] n_r n_s \tag{5.75}$$

or

$$B(\eta_1, \eta_2) = \frac{1}{2} \Gamma_{rs} n_r(\eta_1, \eta_2) n_s(\eta_1, \eta_2) \qquad (r,s = 1, 2, \ldots 8) \tag{5.76}$$

where the result of the thickness integration Γ_{rs} is given in Appendix A.

Using the above form for the complementary energy density in Equation (5.44), one obtains

$$\varepsilon_r = \frac{\partial \Gamma_{rs}}{\partial n_r} n_s \qquad (r, s = 1, 2, \ldots, 8) \tag{5.77}$$

Taking the inverse, one obtains the resultant-reference strain relations

$$n_r = E_{rs}\, \varepsilon_s \qquad (r, s = 1, 2, \ldots, 8) \tag{5.78}$$

2. <u>Finite Element Formulation</u>

The potential energy for the present numerical formulation is

$$\pi_p = \sum_{n=1}^{N} \iint_{S_n} [\tfrac{1}{2} E_{rs}\, \varepsilon_r\, \varepsilon_s + \tfrac{1}{2} N^\circ_{ij}\, r_i\, r_j - p_i\, r_i] dS$$

$$(r, s = 1, 2, \ldots 8;\ i, j = 1, 2, \ldots 15) \tag{5.79}$$

where N = number of elements, and where the initial stress matrix N°_{ij} and the incremental load matrix p_i are defined in Appendix A.

In Equation (5.79), the vector r_i (i = 1, 15) refers to the generalized nodal displacements based on a linear interpolation scheme for the three rectilinear displacements and for the two rotations.

Thus, the reference surface strains can be put in the form (see Appendix A),

$$\varepsilon_r = B_{rs}(\eta_1, \eta_2) r_s \tag{5.80}$$

so that

$$\pi_p = \sum_{n=1}^{N} [\tfrac{1}{2} \hat{k}_{rs}\, r_r\, r_s - \hat{p}_s\, r_s]_n \quad (r, s = 1, 2, \ldots, 15) \tag{5.81}$$

where the stiffness and load matrices $\hat{k}_{rs}$ and $\hat{p}_s$ in the local frame are

$$\hat{k}_{tq} = \iint_{S_n} [E_{rs} B_{rt} B_{sq} + N^{\circ}_{tq}] d\eta_1 \, d\eta_2$$

and

$$\hat{p}_r = \iint_{S_n} p_r \, d\eta_1 \, d\eta_2 \tag{5.82}$$

For the rotationally symmetric problem, the global coordinate frame is that of the lines of curvature coordinates. Thus, if the transformation matrix from the local to the global frame is T_{rs}, then

$$r_i = T_{ij} \, q_j$$

$$\hat{p}_i = T_{ij} \, p_j \tag{5.83}$$

where q_i and p_j refer to the generalized nodal displacements and nodal loads in the global (lines of curvature) coordinate system. Thus,

$$\pi_p = \sum_{n=1}^{N} [\tfrac{1}{2} k_{rs} \, q_r \, q_s - p_r \, q_r]_n \tag{5.84}$$

where the element stiffness and load matrices in the global frame are

$$k_{rs} = T_{ri} \, \hat{k}_{ij} \, T_{js}$$

$$p_r = T_{rs} \, \hat{p}_s \tag{5.85}$$

The numbering scheme for the generalized nodal displacements is defined by

$$\bar{u}\ (P) = q_1 \frac{\bar{a}_1}{\sqrt{a_{11}}} + q_2 \frac{\bar{a}_2}{\sqrt{a_{22}}} + q_3 \frac{\bar{a}_3}{\sqrt{a_{33}}}$$

$$\bar{\omega}\ (P) = -\ q_5 \frac{\bar{a}_1}{\sqrt{a_{11}}} + \frac{\bar{a}_2}{\sqrt{a_{22}}} q_4 \qquad (5.86)$$

where P is the first node of the triangular element.

After solving the variational equation $\delta\pi_p = 0$ for the unrestrained generalized nodal displacements, the following calculations can easily be performed:

(a) local displacements:

$$r_s = T_{sr}\ q_r \qquad (5.87)$$

(b) reference surface strains:

$$\varepsilon_r = B_{rs}(\eta_1,\ \eta_2) r_s \qquad (5.88)$$

(c) stress resultants:

$$n_r = E_{rs}\ \varepsilon_s(\eta_1,\ \eta_2) \qquad (5.89)$$

(d) layer stresses:

$$\tau_i^k = N_{ij}^k(z)\ n_j(\eta_1,\ \eta_2) \qquad (5.90)$$

(e) update the geometry: in the reference configuration, the nodal position vector is given by Equation (3.1),

$$\bar{r}\ (P) = x_i\ (P)\ \bar{e}_i \qquad (5.91)$$

thus, the new nodal position becomes:

$$\bar{R}\ (P) = \bar{r}\ (P) + \left[\bar{u}\ (P) \cdot \bar{e}_k\right]\bar{e}_k \qquad (5.92)$$

3. Equilibrium Check

As outlined in Section II, the equilibrium check is performed at each increment by retaining the correction term, ε^*, defined by Equation (2.20) in the potential energy formulation. Thus, Equation (5.79) is modified as follows:

$$\pi_p = \sum_{n=1}^{N} \iint_{S_n} [\tfrac{1}{2} E_{rs}\, \varepsilon_r\, \varepsilon_s + \tfrac{1}{2} N_{ij}\, r_i\, r_j - p_i\, r_i] dS + \iint_{S_n} [n_r^0\, \varepsilon_r - p_r^0\, r_r] dS \qquad (5.93)$$

where

$$n_r^0 = \text{column}(n_{11}^0, n_{22}^0, n_{12}^0, m_{11}^0, m_{22}^0, m_{12}^0, r_1^0, r_2^0) \; .$$

and p^0 is the load vector at the previous increment. Using Equation (5.80) for the reference surface strains, the above functionsal becomes

$$\pi_p = \sum_{n=1}^{N} \left[\tfrac{1}{2} \hat{k}_{rs}\, r_r\, r_s - \hat{p}_s\, r_s + \hat{R}_s\, r_s\right]_n \qquad (5.94)$$

where the residual load vector, $\hat{R}_s$, is

$$\hat{R}_s = - \iint_{S_n} (n_r^0\, B_{rs} - p_s^0) dS \qquad (5.95)$$

Thus, the element stiffness equation becomes

$$\hat{k}_{rs}\, r_s = \hat{p}_r + \hat{R}_r \qquad (5.96)$$

where $\hat{R}_r$ is known from the previous step and $\hat{p}_r$ is the current incremental load.

Next, Equation (5.96) is evaluated for each element and by applying interelement compatibility, these can be assembled into a system of linear incremental equilibrium equations for the entire structure,

$$K^N q_N = Q_N + R_N \tag{5.97}$$

where

K^N = incremental stiffness at the N^{th} step,

q_N = incremental nodal displacements,

Q_N = current incremental load, and

R_N = residual load.

If the N-1 state is exactly in equilibrium, then the residual load, R_N is zero.

For the first incremental step (linear problem) $R_1 = 0$, so that

$$q_1 = (K^1)^{-1} Q_1 \tag{5.98}$$

After this and every succeeding step, the geometrical configuration is updated and the total stresses and strains are calculated by adding all incremental contributions, and a new stiffness matrix is formed.

As shown in Figure 11, $q_1 \neq q_1^*$, where q_1^* represents the true solution. Thus, the residual load, R_2, is calculated using the solution q_1, and at the second step the following system of linear equations is solved:

$$K^2 q_2 = Q_2 + R_2 \tag{5.99}$$

The quantity, R_2, represents the unbalance in nodal equilibrium, introduced by the linearization process. This process is illustrated by Figure 11.

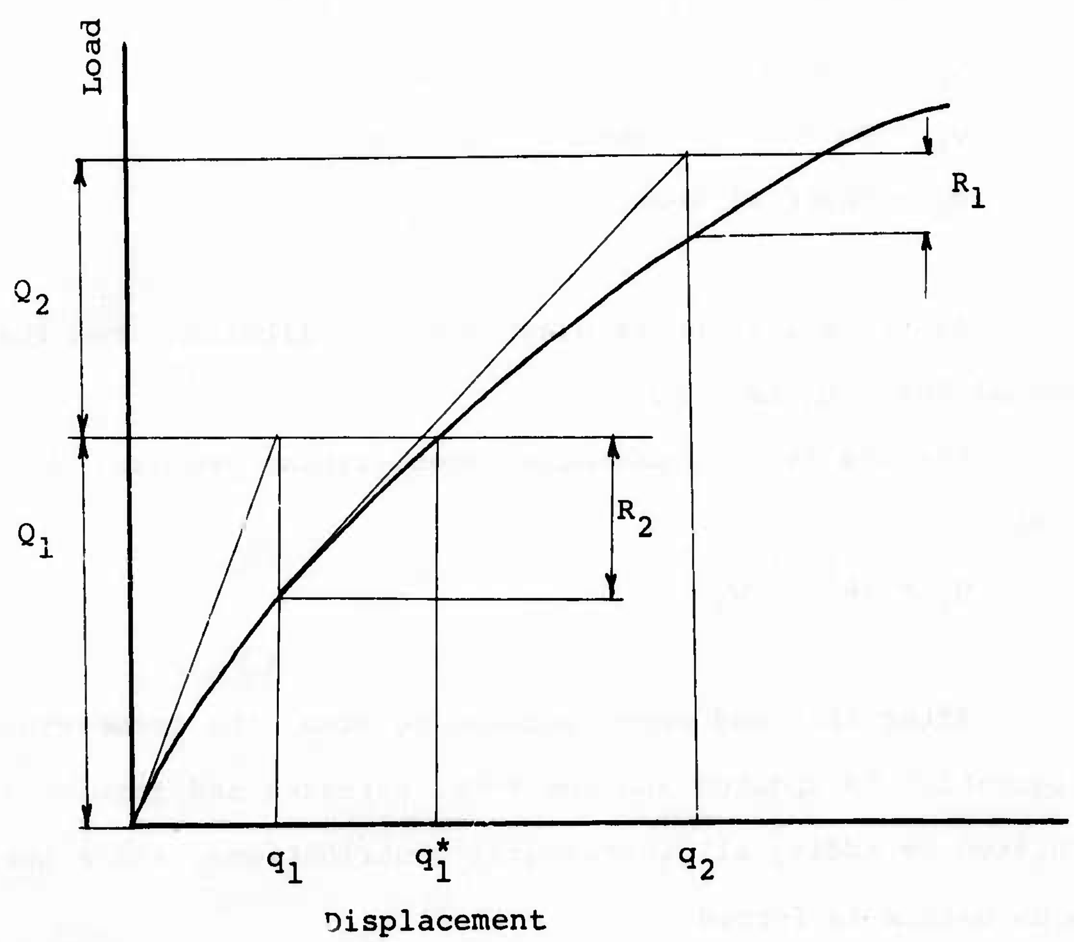


Figure 11. Incremental Process

D. Numerical Results

First, a clamped square plate under uniform pressure is considered. The variation of the central deflection with respect to the pressure load is shown in Figure 12. The exact solution indicated in this figure does not take shear deformation into account.

The next example is that of the inflation of a six-ply aircraft tire [7]. Uniform thickness is assumed with the following material characteristics:

Cord Properties:

$E_c = 1.56 \times 10^5$ psi,

$\nu_c = 0.5$ psi, and

d (cord diameter) = 0.031 in.

Rubber Properties:

$E_r = 450$ psi, and

$\nu_r = 0.49$.

Tire Construction Parameters:

n (number of plies) = 6,

Green angle = 57 degrees,

bead radius = 9.15 in.,

number of cords per inch = 26, and

ply thickness = 0.043 in.

The undeformed meridian profile is depicted by Figure 13 and also described in a cartesion reference frame in Table 1.

The initial cord angle varies with the meridian position according to the classical lift equation of bias tire constructions, whose results are shown in Figure 14.

In the numerical solution, rotational symmetry is employed by specifying zero rotation with respect to the meridian and restraining the motion in the parallel direction of a discretized strip consisting of 80 elements and 62 nodes, as shown in Figure 15.

The variation of the crown displacement with respect to inflation pressure is depicted in Figure 16, showing a substantial deviation from that of the exact numerical solution of [7], which does not consider shear deformation. Therefore, further verification is required to evaluate the presented numerical results.

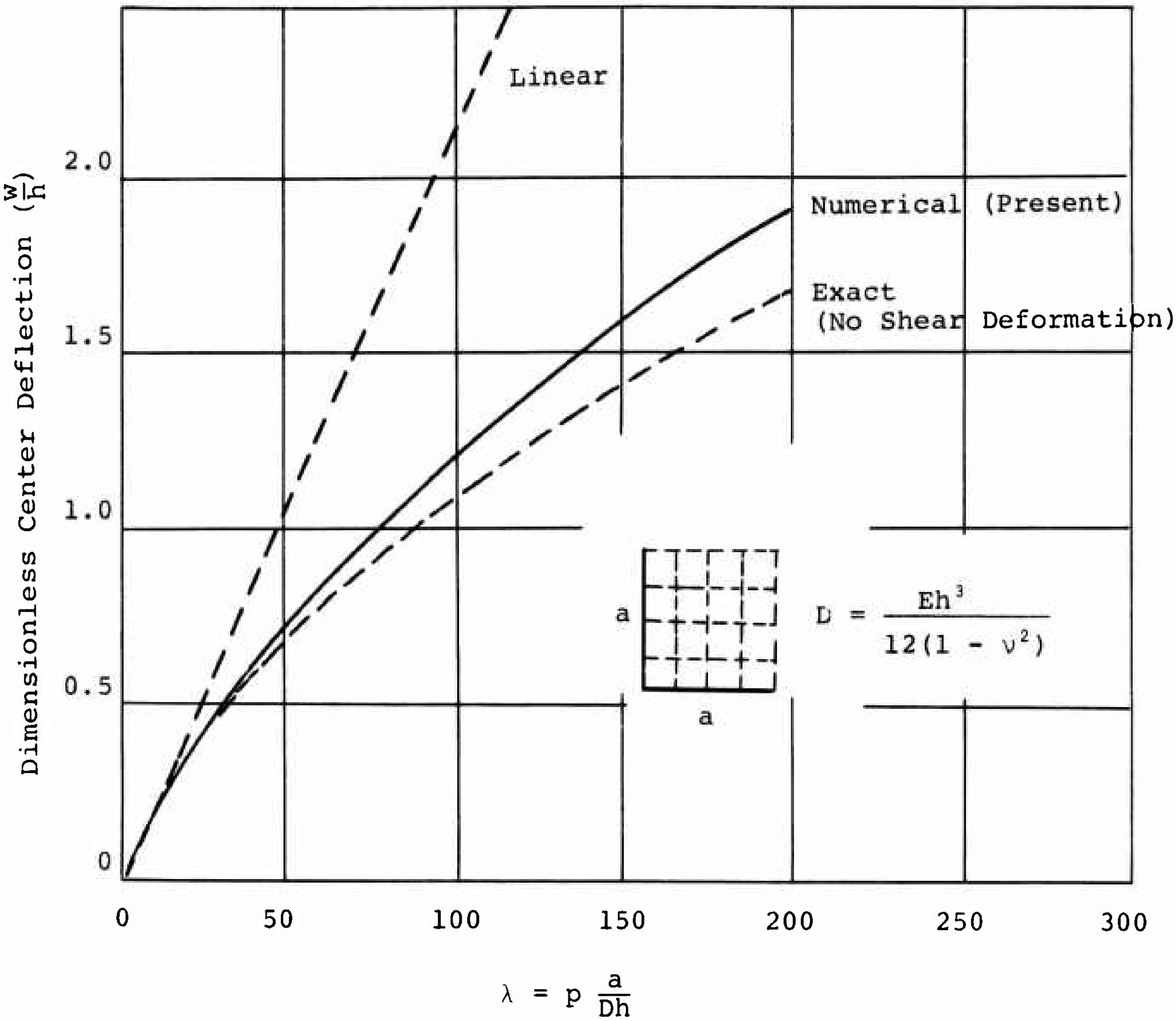


Figure 12. Center Deflection versus Pressure Loading for a Square Clamped Plate

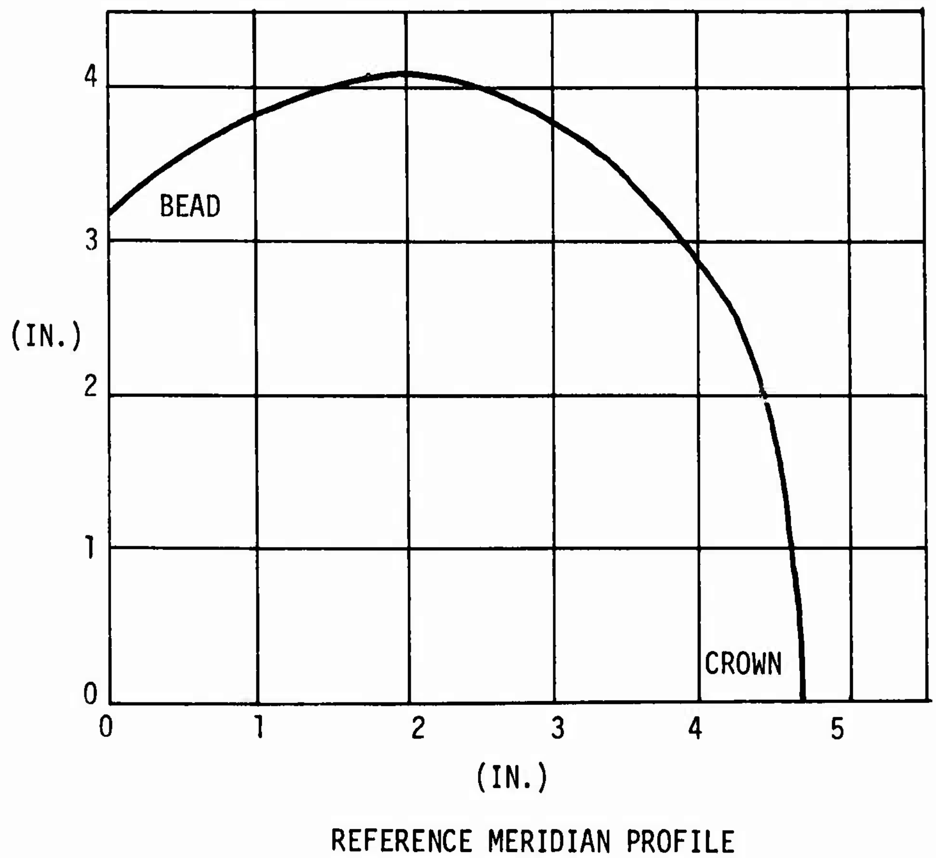


Figure 13. Undeformed Meridian Reference Surface

Table 1

Meridian Profile Description

x	y
4.69	0.00
4.68	0.37
4.66	0.74
4.63	1.10
4.58	1.47
4.51	1.84
4.40	2.19
4.25	2.52
4.05	2.84
3.83	3.13
3.57	3.39
3.29	3.62
2.97	3.81
2.63	3.95
2.29	4.03
1.89	4.04
1.52	3.97
1.16	3.88
0.82	3.73
0.50	3.55
0.18	3.35
0.00	3.24

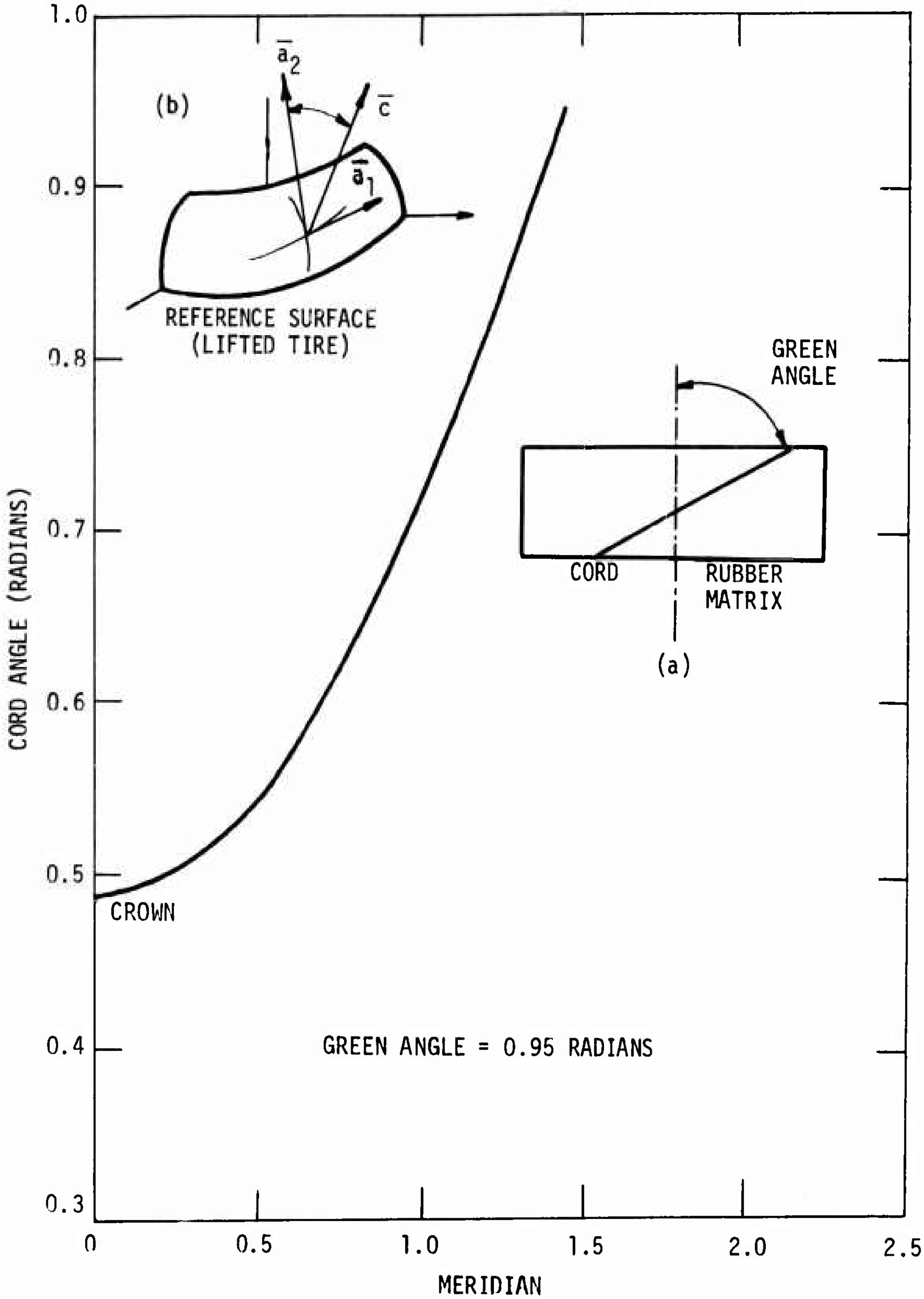


Figure 14. Cord Angle Distribution

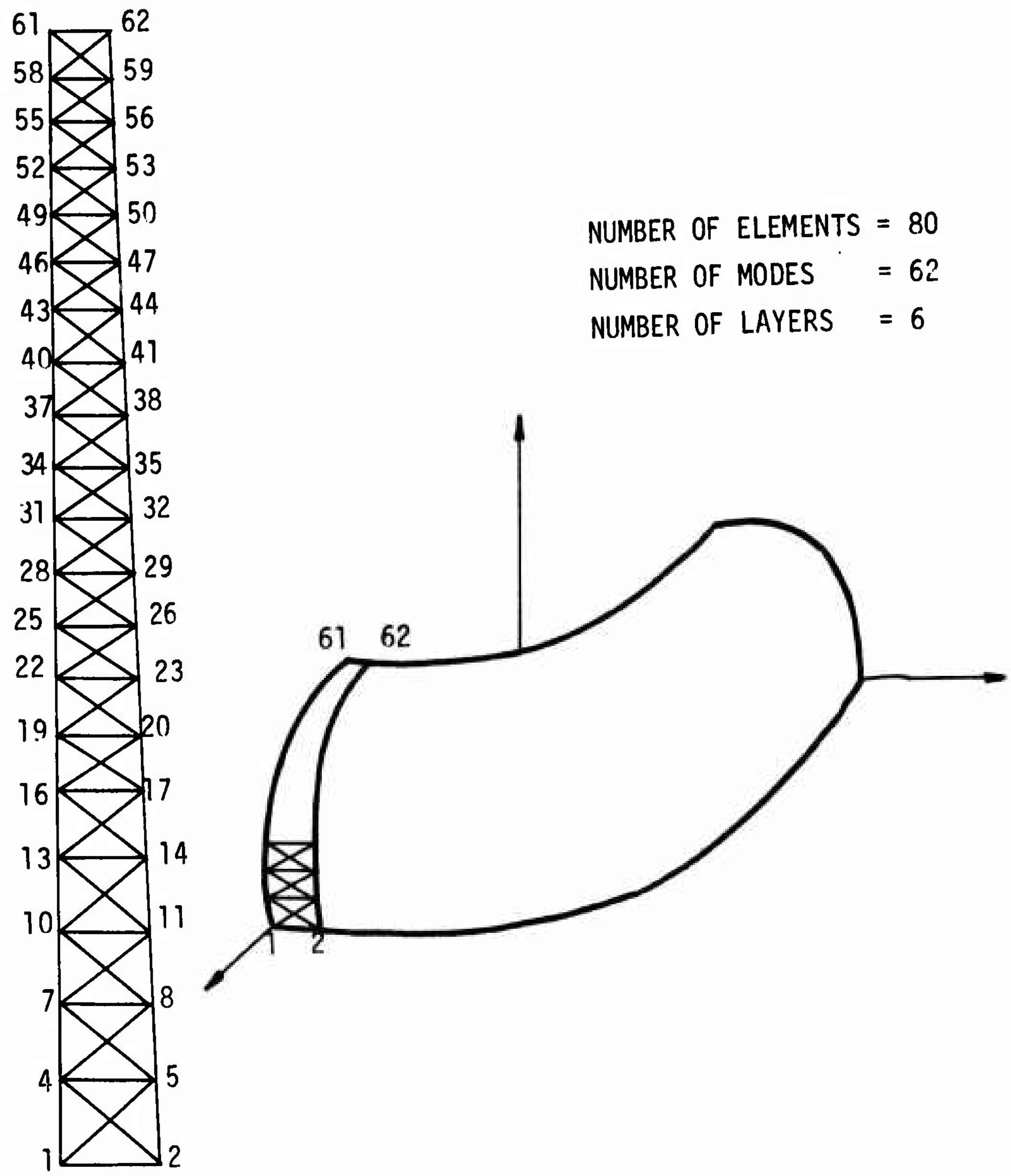


Figure 15. Strip Discretization Along the Meridian

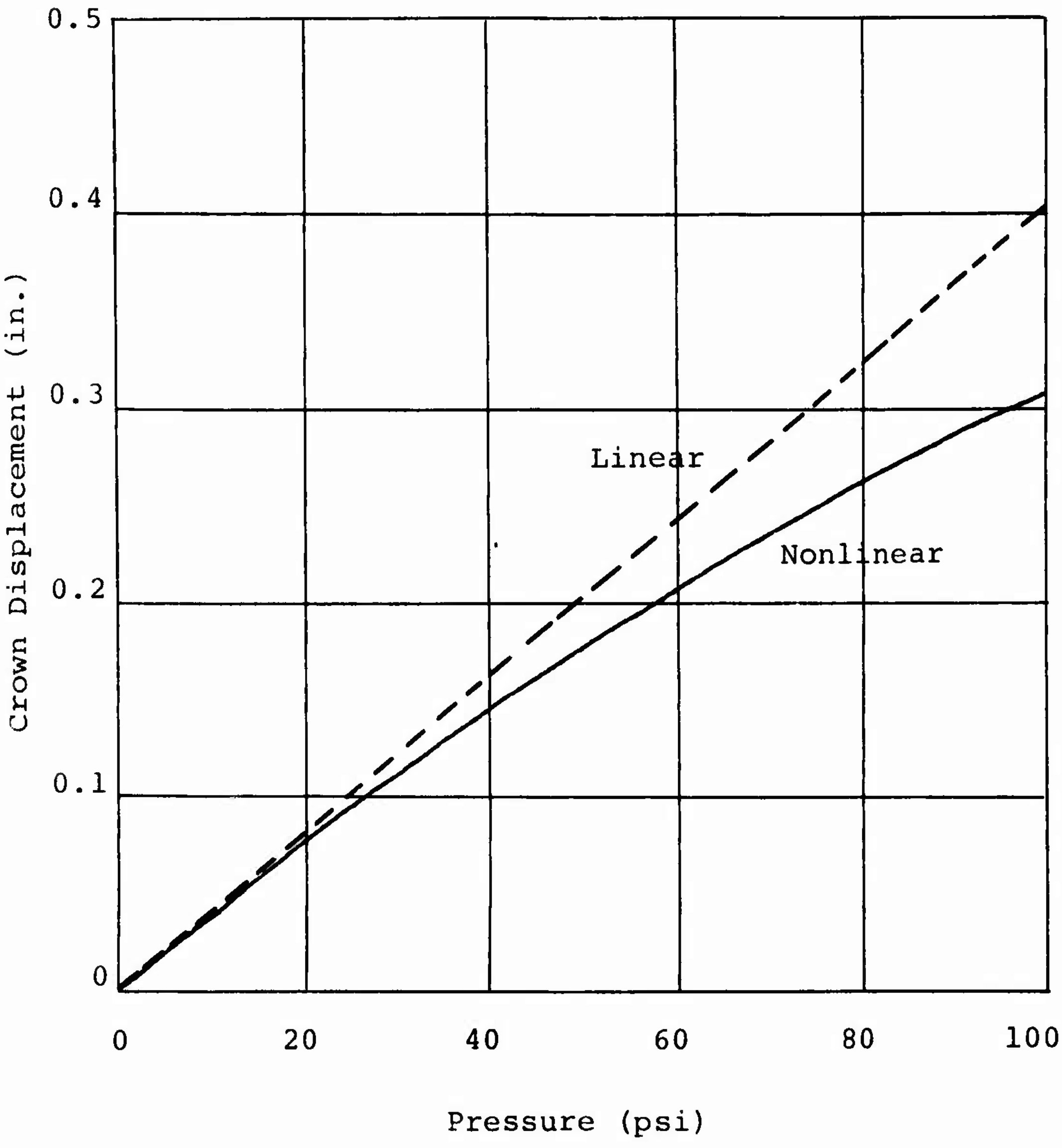


Figure 16. Crown Displacement versus Inflation Pressure

APPENDIX A

ATTACHMENT TO NUMERICAL FORMULATION

Equation (5.74):

$$a = 2\, s_k/A; \qquad b = 2\, s_k z/B; \qquad c = -\frac{1}{B}\left[s_k(z^2 - c_k^2) + S_k\right]$$

$$N_{11}^k = a \qquad N_{14}^k = b$$

$$N_{22}^k = a \qquad N_{25}^k = b$$

$$N_{33}^k = a \qquad N_{26}^k = b$$

$$N_{47}^k = c \qquad N_{58}^k = c \tag{A.1}$$

and all others are zero.

Equation (5.76):

$$\alpha_1^k = 4\, s_k^2\, (c_{k+1} - c_k)/A^2$$

$$\alpha_2^k = 2\, s_k^2\, (c_{k+1}^2 - c_k^2)/(AB)$$

$$\alpha_3^k = \frac{4}{3} s_k^2\, (c_{k+1}^3 - c_k^3)/B$$

$$\begin{aligned}\alpha_4^k = \{ & s_k^2\, [(c_{k+1}^5 - c_k^5)/5 - \frac{2}{3} c_k^2(c_{k+1}^3 - c_k^3) + c_k^4(c_{k+1} - c_k)] \\ & + S_k^2(c_{k+1} - c_k) + 2\, s_k\, S_k[(c_{k+1}^3 - c_k^3)/3 \\ & - c_k^2(c_{k+1} - c_k)]\}/B^2\end{aligned} \tag{A.2}$$

From Equation (5.79), one defines

$$\begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{bmatrix}^k = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{12} & c_{22} & c_{23} \\ c_{13} & c_{23} & c_{33} \end{bmatrix}^k \begin{bmatrix} \tau_{11} \\ \tau_{22} \\ \tau_{12} \end{bmatrix}^k \tag{A.3}$$

and

$$\begin{bmatrix} \gamma_{13} \\ \gamma_{23} \end{bmatrix}^k = \begin{bmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{bmatrix}^k \begin{bmatrix} \tau_{13} \\ \tau_{23} \end{bmatrix}^k \tag{A.4}$$

where the coefficients matrices will be denoted by C^k_{rs} and S^k_{rs}, respectively. Then, it follows that

$$\Gamma_{rs} = \sum_{k=1}^{L} \alpha_1^k C^k_{rs}$$

$$\Gamma_{r,s+3} = \sum_{k=1}^{L} \alpha_2^k C^k_{rs}$$

$$\Gamma_{r+3,s} = \Gamma_{s,r+3}$$

$$\Gamma_{r+3,s+3} = \sum_{k=1}^{L} \alpha_3^k C^k_{rs} \qquad (r,s = 1, 2, 3) \tag{A.5}$$

and

$$\Gamma_{r+6,s+6} = \sum_{k=1}^{L} \alpha_4^k S^k_{rs} \qquad (r,s = 1,2) \tag{A.6}$$

and all others are zero.

Equation (5.80):

In a local barycentric coordinate system, the membrane displacements (u_1, u_2), the rotations (ω_1, ω_2), and the normal displacement (w) are linearly interpolated. Thus, the components of the reference surface strain tensor are

$$\varepsilon_i = F_{ij} \beta_j \qquad (i = 1,2,\ldots 8;\ j = 1,2,\ldots 12) \tag{A.7}$$

where

$$\varepsilon_i = \text{column}\ (\varepsilon_{11}, \varepsilon_{22}, \gamma_{12}, k_{11}, k_{22}, 2\,k_{12}, \gamma_{13}, \gamma_{23}) \tag{A.8}$$

$$\begin{array}{llll} F_{11} = 1 & F_{24} = 1 & F_{32} = 1 & F_{33} = 1 \\ F_{46} = 1 & F_{5,10} = 1 & F_{6,7} = 1 & F_{6,9} = 1 \\ F_{7,5} = 1 & F_{7,6} = \eta_1 & F_{7,7} = \eta_2 & F_{7,11} = 1 \\ F_{8,8} = 1 & F_{8,9} = \eta_1 & F_{8,10} = \eta_2 & F_{8,12} = 1 \end{array} \tag{A.9}$$

The vector β_i is defined by

$$u_1 = a_1 + \beta_1 \eta_1 + \beta_2 \eta_2$$

$$u_2 = b_1 + \beta_3 \eta_1 + \beta_4 \eta_2$$

$$\omega_1 = \beta_5 + \beta_6 \eta_1 + \beta_7 \eta_2$$

$$\omega_2 = \beta_8 + \beta_9\,\eta_1 + \beta_{10}\,\eta_2$$

$$w = c_1 + \beta_{11}\,\eta_1 + \beta_{12}\,\eta_2$$

In the local frame, the generalized nodal displacement schedule is

$$\begin{array}{lll}
u_1(P) = r_1 & u_1(Q) = r_2 & u_1(R) = r_3 \\
u_2(P) = r_4 & u_2(Q) = r_5 & u_2(R) = r_6 \\
\omega_1(P) = r_7 & \omega_1(Q) = r_8 & \omega_1(R) = r_9 \\
\omega_2(P) = r_{10} & \omega_2(Q) = r_{11} & \omega_2(R) = r_{12} \\
w(P) = r_{13} & w(Q) = r_{14} & w(R) = r_{15}
\end{array} \qquad \text{(A.11)}$$

The result of the interpolation according to Equations (A.10) and (A.11) is

$$\beta_i = R_{ij}\,r_j \qquad (i = 1,2,\ldots 12;\; j = 1,2,\ldots 15) \qquad \text{(A.12)}$$

Thus, Equation (A.7) becomes

$$\varepsilon_i = F_{ij}\,R_{js}\,r_s \qquad \text{(A.13)}$$

so that in Equation (5.80),

$$B_{rs}(\eta_1, \eta_2) = F_{rk}(\eta_1, \eta_2)R_{ks} \qquad \text{(A.14)}$$

Equation (5.79):

The initial stress matrix in Equation (5.79) is calculated from

$$c = n^{\circ}_{\alpha\beta}\, w_{,\alpha}\, w_{,\beta} \tag{A.15}$$

so that

$$N^{\circ}_{rs} = R^{T}_{rp}\, n^{\circ}_{pk}\, R_{ks} \qquad (r,s = 1,2,\ldots 15;\ p,k = 1,2,\ldots 12) \tag{A.16}$$

where

$$n^{\circ}_{11,11} = n^{\circ}_{11}$$

$$n^{\circ}_{11,12} = n^{\circ}_{12}$$

$$n^{\circ}_{12,11} = n^{\circ}_{12}$$

$$n^{\circ}_{12,12} = n^{\circ}_{22} \tag{A.17}$$

and all others are zero.

The load matrix in Equation (5.79) is derived as follows. For constant pressure loading, consider

$$pw(\eta_1, \eta_2) = p(c_1 + \beta_{11}\,\eta_1 + \beta_{12}\,\eta_2) \tag{A.18}$$

In a barycentric coordinate system, it is sufficient to consider

$$pw(\eta_1, \eta_2) = pc_1 \tag{A.19}$$

where c_1 is determined from a linear interpolation in the form

$$c_1 = I_{11}\, r_{13} + I_{12}\, r_{14} + I_{13}\, r_{15} \tag{A.20}$$

Thus, the components of the pressure load matrix are

$$p_{13} = p\, I_{11}$$

$$p_{14} = p\, I_{12}$$

$$p_{15} = p\, I_{13} \tag{A.21}$$

For the derivation of the inertial load matrix, begin with Equation (3.43),

$$\bar{f} = [\, - \Omega^2\, x_\alpha \int_{(h)} \gamma(\xi_3) d\, \xi_3]\bar{e}_\alpha \tag{A.22}$$

The mass density in the above equation is calculated for each layer from the cord and rubber mass densities as

$$\gamma_k = r_k\, \gamma_k^C + (1 - r_k)\gamma_k^R \tag{A.23}$$

where r_k is defined by Equation (4.10). Thus, the total mass density m is

$$m \equiv \int_{(h)} \gamma(\xi_3) d\, \xi_3 = \sum_{k=1}^{L} \gamma_k (c_{k+1} - c_k) \tag{A.24}$$

which is concentrated at the centroid of each element, so that

$$\bar{f} = -\, m\, \Omega^2\, \hat{x}_\alpha\, \bar{e}_\alpha \tag{A.25}$$

The corresponding inertial load matrix is then defined by

$$\bar{f} \cdot [u_1(\eta_1, \eta_2)\bar{g}_1 + u_2(\eta_1, \eta_2)\bar{g}_2 + w(\eta_1, \eta_2)\bar{g}_3] = p_i \; r_i \qquad \text{(A.26)}$$

Therefore, in a barycentric coordinate system,

$$p_1 = - m \, \Omega^2 \, I_{11} \, \hat{x}_\alpha \, (\bar{e}_\alpha \cdot \bar{g}_1)$$

$$p_2 = - m \, \Omega^2 \, I_{12} \, \hat{x}_\alpha \, (\bar{e}_\alpha \cdot \bar{g}_1)$$

$$p_3 = - m \, \Omega^2 \, I_{13} \, \hat{x}_\alpha \, (\bar{e}_\alpha \cdot \bar{g}_1)$$

$$p_4 = - m \, \Omega^2 \, I_{11} \, \hat{x}_\alpha \, (\bar{e}_\alpha \cdot \bar{g}_2)$$

$$p_5 = - m \, \Omega^2 \, I_{12} \, \hat{x}_\alpha \, (\bar{e}_\alpha \cdot \bar{g}_2)$$

$$p_6 = - m \, \Omega^2 \, I_{13} \, \hat{x}_\alpha \, (\bar{e}_\alpha \cdot \bar{g}_2)$$

$$p_{13} = - m \, \Omega^2 \, I_{11} \, \hat{x}_\alpha \, (\bar{e}_\alpha \cdot \bar{g}_3)$$

$$p_{14} = - m \, \Omega^2 \, I_{12} \, \hat{x}_\alpha \, (\bar{e}_\alpha \cdot \bar{g}_3)$$

$$p_{15} = - m \, \Omega^2 \, I_{13} \, \hat{x}_\alpha \, (\bar{e}_\alpha \cdot \bar{g}_3) \qquad \text{(A.27)}$$

APPENDIX B

CLOSED FORM PLATE SOLUTION

For comparison purposes, an analytical solution is derived here for the response of a moderately thick, simple supported plate under sinusoidal pressure loading.

From Section II, the field equations of an isotropic plate are as follows.

Equilibrium equations:

$$m_{\alpha\beta,\beta} - r_\alpha = 0 \tag{B.1}$$

$$r_{\alpha,\alpha} + p = 0 \tag{B.2}$$

Resultant-displacement and rotation gradient relations:

$$m_{11} = D(\omega_{1,1} + \nu\, \omega_{2,2})$$

$$m_{22} = D(\omega_{2,2} + \nu\, \omega_{1,1})$$

$$m_{12} = D(1 - \nu)(\omega_{1,2} + \omega_{2,1})/2 \tag{B.3}$$

$$r_1 = 5/h^2\, D(1 - \nu)(w_{,1} + \omega_1)$$

$$r_2 = 5/h^2\, D(1 - \nu)(w_{,2} + \omega_2) \tag{B.4}$$

where

$$D = \frac{E\, h^3}{12(1 - \nu^2)} \tag{B.5}$$

The solution of the above set of equations may easily be obtained by the following change in variables:

$$\Phi = \omega_{1,2} - \omega_{2,1} \quad \text{(definition)}$$

$$\Theta = \omega_{1,1} + \omega_{2,2} \quad \text{(definition)} \qquad \text{(B.6)}$$

It can be shown that the governing equations are

$$D\,\nabla^2 \Theta + p = 0$$

$$\nabla^2 \Phi = 10/h^2\,\Phi$$

$$5/h^2\,D(1 - \nu)(\nabla^2 w + \Theta) + p = 0 \qquad \text{(B.7)}$$

Consider now a rectangular plate of dimensions a x b, such that the edges are defined by $x_1 = \pm a/2$ and $x_2 = \pm b/2$. Let the pressure load be defined by

$$p = p \cos \alpha x_1 \cos \beta x_2$$

$$\alpha = \pi/a$$

$$\beta = \pi/b \qquad \text{(B.8)}$$

In a simple supported situation, Φ is identically zero and w and Θ have the forms

$$w = w_0 \cos \alpha x_1 \cos \beta x_2$$

$$\Theta = \Theta_0 \cos \alpha x_1 \cos \beta x_2 \qquad \text{(B.9)}$$

where

$$w_o = p_o/D\,(1 + k)\,\frac{1}{(\alpha^2 + \beta^2)^2}$$

$$\theta_o = p_o/D\,\frac{1}{\alpha^2 + \beta^2}$$

and

$$k = (\alpha^2 + \beta^2)h^2/5(1 - \nu) \tag{B.10}$$

The rotations and shear strains then become

$$\omega_1 = \frac{p_o\,\alpha}{D(\alpha^2 + \beta^2)^2}\ \sin\alpha x_1 \cos\beta x_2$$

$$\omega_2 = \frac{p_o\,\alpha}{D(\alpha^2 + \beta^2)^2}\ \cos\alpha x_1 \sin\beta x_2$$

$$\gamma_1 = -\frac{k\,\alpha\,p_o}{D(\alpha^2 + \beta^2)^2}\ \sin\alpha x_1 \cos\beta x_2$$

$$\gamma_2 = -\frac{k\,\beta\,p_o}{D(\alpha^2 + \beta^2)}\ \cos\alpha x_1 \sin\beta x_2 \tag{B.11}$$

so that the $k = 0$ situation corresponds to the Kirchoff-Love plate theory.

The resultants become

$$m_{11} = p_o(\alpha^2 + \nu\,\beta^2)/(\alpha^2 + \beta^2)^2 \cos\alpha x_1 \cos\beta x_2$$

$$m_{22} = p_o(\nu\,\alpha^2 + \beta^2)/(\alpha^2 + \beta^2)^2 \cos\alpha x_1 \cos\beta x_2$$

$$m_{12} = -\,p_o(1 - \nu)\alpha\beta/(\alpha^2 + \beta^2)^2 \sin\alpha x_1 \sin\beta x_2$$

$$r_1 = - p_0 \, \alpha/(\alpha^2 + \beta^2) \sin \alpha x_1 \cos \beta x_2$$

$$r_2 = - p_0 \, \beta/(\alpha^2 + \beta^2) \cos \alpha x_1 \sin \beta x_2 \qquad (B.12)$$

REFERENCES

1. T. H. H. Pian, "Derivation of Element Stiffness Matrices by Assumed Stress Distributions", AIAA J. 2:1333-1336 (1964), No. 7.

2. T. H. H. Pian and P. Tong, "Rationalization in Receiving Element Stiffness Matrix by Assumed Stress Approach", Proc. 2nd Conference in Structural Mechanics, AFFDL-HR-68-150, WPAFB, Ohio, 1968, pp. 461-471.

3. T. H. H. Pian, "Hybrid Models", paper presented at the International Symposium on Numerical and Computer Methods in Structural Mechanics, University of Illinois at Urbana, Champaign, September 8-10, 1971.

4. T. H. H. Pian and P. Tong, "Basis of Finite Element Method for Solid Continua", Int. J. Numerical Methods in Eng. 1:3-28 (1969), No. 1.

5. A. L. Deak and S. Atluri, "Nonlinear Stress Analysis of Laminated Shells by the Hybrid Stress Method", presented at the 1974 International Conference on Computational Methods in Nonlinear Mechanics, Austin, Texas, September, 1974.

6. S. Atluri, "On the Hybrid Stress Finite Element Model for Incremental Analysis of Large Deflection Problems", Intl. J. Solids Structures 9:1177-1191 (1973).

7. H. K. Brewer, "Stress and Deformations in Multi-Ply Aircraft Tires Subject to Inflation Pressure Loading", AFFDL-TR-70-62, June 1970.

8. L. D. Hoffmeister, G. A. Greenbaum, and D. A. Evensen, "Large Strain, Elastic-Plastic Finite Element Analysis", AIAA J. 9:1248-1255 (1971), No. 7.

9. K. Wasizu, Variational Methods in Elasticity and Plasticity (Pergamon Press, New York, 1968).

10. A. E. Green and W. Zerna, Theoretical Elasticity (At the Clarendon Press, Oxford, 1954).

11. V. V. Novozhilov, Thin Shell Theory (P. Noordhoff Ltd.-Groningen, The Netherlands, 1964).

12. S. G. Lekhnitskii, Theory of Elasticity of an Anisotropic Elastic Body (Holden Day, Inc., San Francisco, 1963).

13. C. Truesdell and W. Noll, The Nonlinear Field Theories of Mechanics (Springer-Verlag, Berlin, 1965).

14. A. H. Stroud and D. Secrest, Gaussian Quadrature Formulas (Prentice-Hall, Inc., Englewood Cliffs, 1966).

☆U.S.Government Printing Office: 1977 — 757-001/15